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CHAPTER I

POINTS OF VIEW

THE OBJECT OF MATHEMATICS

"MATHEMATICS is queen of the sciences and arithmetic the queen of mathematics. She often condescends to render service to astronomy and other natural sciences, but under all circumstances the first place is her due."

So wrote the master mathematician, astronomer, and physicist, Gauss (1777–1855) over a century ago. Whether as history or prophecy, Gauss' declaration is far from an overstatement. Time after time in the nineteenth and twentieth centuries, major scientific theories have come into being only because the very ideas in terms of which the theories have meaning were created by mathematicians years, or decades, or even centuries before anyone foresaw possible applications to science.

Without the geometry of Riemann, published in 1854, or without the theory of invariance developed by the mathematicians Cayley (1821–1895), Sylvester (1814–1897), and a host of their followers, the general theory of relativity and gravitation of Einstein in 1916 could not have been stated. Without the whole mathematical theory of boundary value problems—to use a technical term which need not be explained now—originating with Sturm (1803–1855) and Liouville (1809–1882), the far-reaching wave mechanics of the atom of the past five years would have been impossible.

The revolution in modern physics which began with the work of W. Heisenberg and P. A. M. Dirac in 1926 could never have started without the necessary mathematics of matrices invented by Cayley in 1858, and elaborated by a small army of mathematicians from then to the present time.

The concept of invariance, of that which remains unchanged in the ceaseless flux of nature, permeates modern physics, and it originated in 1801 in the purely arithmetical work of Gauss.

These are but a few of many similar instances. In none of the scores of anticipations of fruitful applications to science was there any thought of what might come out of the pure mathematics. Guided only by their feeling for symmetry, simplicity, and generality, and an indefinable sense of the fitness of things, creative mathematicians now as in the past are inspired by the art of mathematics rather than by any prospect of ultimate usefulness. However it may be in engineering and the sciences, in mathematics the deliberate attempt to create something of immediate utility leads as a rule to shoddy work of only passing value. The important practical and scientific applications of mathematics are unsought byproducts of the main purposes of professional mathematicians.

The queen of the sciences however needs no shabby apology as an introduction. Jacobi (1804–1851) fittingly expressed what many believe to be the true purpose of mathematics in his retort to Fourier (1768–1830). To appreciate this we must recall that Fourier's influence on pure mathematics is comparable to Jacobi's on applied mathematics. In his analytical theory of heat (published in 1822), the applied mathematician Fourier devised tools which are as useful today in pure mathematics as they are in all physics where wave motion underlies the pattern of events. On the other hand, the contributions of the pure mathematician Jacobi to higher mechanics are indispensable in modern physics. Fourier had reproached Jacobi for "trifling with pure mathematics." Jacobi replied that a scientist of Fourier's calibre should know that the true end of mathematics is the greater glory of the human mind.

A GOLDEN AGE

In the past hundred years mathematics entered its golden age. This most prolific period in the history of mathematics had well started by 1830; the end is not yet in sight. No previous age approaches the past century for the depth and tremendous sweep of its mathematics. The only other centuries at all comparable with the past hundred years are those of Archimedes (287–212 B.C.) and Newton (1642–1727), and these can be compared with the Century of Progress only when generous allowance is made for the difficulties of pioneering. The mathematical inheritance of the past century from its predecessors was great, both in quantity and quality, so great indeed that one prophet in 1830 lamented that "the golden age of mathematical literature is undoubtedly past." That splendid inheritance of at least twenty centuries was increased many times in one hundred years.

So vast has been the increase of mathematical knowledge in the past century that few men would presume to claim more than an amateur's acquaintance with more than one of the four major divisions of modern mathematics. The field of higher arithmetic alone as it exists today is probably beyond the complete mastery of any two men, while geometry, algebra and analysis, especially the last, are of even greater extent. If mathematical physics be annexed as a province of mathematics, a detailed, professional mastery of the whole domain of modern mathematics would demand the lifelong toil of twenty or more richly gifted men.

In all this there is a crumb of comfort for those whose mathematical training ended with their last year in high school or their first year in college. These are not so much worse off, relatively, than the majority of mathematicians who turn the pages of the current mathematical periodicals or attend scientific meetings. Out of fifty mathematical papers presented in brief at such a meeting, it is a rare mathematician indeed who really understands what more than half a dozen are about. The very language in which most of the other forty-four are presented goes clean over the head of the man who follows the six reports nearest his own specialty.

Many causes contribute to this state of affairs which seems to be a necessary consequence of mathematical progress. We need mention only one. It is the perennial youthfulness of mathematics itself which marks it off with a disconcerting immortality from the sciences.

In theoretical physics it is but seldom necessary to master in detail a work published over thirty years ago, or even to remember that such a work was ever written. But in mathematics the man who is ignorant of what Pythagoras said in Croton in 500 B.C. about the square on the longest side of a right angled triangle, or who forgets what someone in Czecho Slovakia proved last week about inequalities, is likely to be lost. The whole terrific mass of well established mathematics, from the ancient Babylonians to the modern Japanese, is as good today as it ever was.

Looking down and far out over the past from our vantage points of today we can only marvel at the dogged courage and persistence of the explorers who first won a devious way through the wilderness. Broad highways now cross the barren deserts, straight as bowstrings; where scores perished miserably in the pestilent marshes there is a thriving city, and the pass through the iron mountains which our forefathers sought in vain is an easy four hours' pleasure trip from the distant city. The loftier range behind the one on which we stand is now accessible to us, although the way is hard, and by scaling its lesser peaks we can catch glimpses of an El Dorado of which the most daring of the pioneers never dreamed.

If we marvel at the patience and the courage of the pioneers, we must also marvel at their persistent blindness in missing the easier ways through the wilderness and over the mountains. What human perversity made them turn east to perish in the desert, when by going west they could have marched straight through to ease and plenty? This is our question today. If there is any continuity in the progress of mathematics, the men of a hundred years hence will be asking the same question about us. We know that there is a higher range behind us, and we suspect that behind that one is a higher, and so on, for as far and as long as there shall be human beings with the spirit of adventure to heed the whisper of the unknown. At the present rate of progress our vantage points of today will be barely distinguishable hillocks in a boundless plain to the explorers of a century hence. Before standing on one or two of the hard-won peaks of the past century to see what we can of the progress made in the last hundred years, let us look about us well before we start.

ABEL'S ADVICE

To get some sort of a perspective, let us consider roughly the kind of mathematics acquired by a student who takes all that is offered in a good American high school. The geometry taught is practically that of Euclid and is 2200 years old.

It is a satisfactory first approximation to the geometry of the physical universe, and it is good enough for engineers, but it is not that which is of vital interest in modern physics, and its interest for working mathematicians evaporated long ago. Our vision of the universe has swept far beyond the geometry of Euclid.

In algebra the case is a little better. A well taught student will master the binomial theorem for a positive whole number exponent which Blaise Pascal discovered in 1653. There he will stop. And yet the really interesting things in algebra are the creation of the Nineteenth and Twentieth Centuries, and began to be developed over a century and a half after Pascal died.

Of higher arithmetic the graduate of a good school will learn precisely nothing. Unless extremely fortunate, he will never even have heard of the theory of numbers. And yet at least one of its most beautiful and far reaching truths was known to Euclid. Many of the most striking results in this field are accessible to anyone with a year of high school training.

In analytical geometry and the calculus the score is again zero. The calculus, however, which has been estimated as the most powerful instrument ever devised for scientific thought, may become part of the regular high school course before the next Chicago World's Fair.

Without a good working knowledge of the differential and integral calculus created by Newton and Leibnitz in the Seventeenth Century it is impossible even to read serious works on the physical sciences and their applications, much less to take a step ahead. The like is true, but to a far lesser extent, for some branches of biology and psychology, and it is beginning to be true for some economics. Any normal boy or girl of sixteen could master the calculus in half the time often devoted to stumbling through Book One of Caesar's Gallic War. And it does seem to some modern minds that Newton and Leibnitz were more inspiring leaders than Julius Caesar and his unimaginative lieutenant Titus Labienus.

The junior college student will be considerably farther ahead at the end of his fourth year. Provided he has not sought culture by the literary trail exclusively, he may be able to appreciate some of the minor classics of science. He will know as much as the men of the Eighteenth Century knew of the calculus, and he will know it better than they did. Much of what passed for proof with the pioneers would not now be tolerated in a college text book. To this light extent the profound critical work of the Nineteenth Century mathematicians has influenced the thinking of those who take the calculus in college—at least in a good college under a man who is not hopelessly dry and dusty.

Before quitting this somewhat uninspiring prospect, let us glance at another of the reasons why the average graduate of standard four year college course in mathematics usually manages to miss completely the spirit of modern mathematics. The point is obvious from a remark of Abel (1802–1829), one of the greatest mathematical geniuses of all time. In his wretched life of less than twenty-seven years Abel accomplished so much of the highest order that one of the leading mathematicians of the Nineteenth Century (Hermite, 1822–1901) could say without exaggeration, "Abel has left mathematicians enough to keep them busy for five hundred years." Asked how he had done all this in the six or seven years of his working life, Abel replied "By studying the masters, not the pupils."

To appreciate the living spirit rather than the dry bones of mathematics it is necessary to inspect the work of a master at first hand. Text books and treatises are a very necessary evil. The mere bulk of the work to be assimilated in any reasonable time precludes intimate contact with the creators through their works. Nevertheless it is not impossible in the ordinary course of education to read at least ten or twenty pages of mathematics as it came from the pen of a master. The very crudities of the first attack on a significant problem by a master are more illuminating than all the pretty elegance of the standard texts which has been won at the cost of perhaps centuries of finicky polishing.

It is rarely feasible for beginners to attempt the mastery of recent work. This appears in the mathematical journals, of which there are now about 500 published throughout the world. Some come out monthly, others quarterly, and the contents of about 200 are almost exclusively accounts of current mathematical research. Most of the articles are in English, Italian, French, or German (particularly the last two), although many are printed in the native languages of the authors, which range from Japanese, Russian, and Polish to Czech and Roumanian. For a competence in modern mathematics a reading knowledge of the first four languages named is a necessity.

Instead of trying to touch the spirit of modern mathematics through any of this up-to-the-minute work, it is much more practicable to study attentively some older classic. Many of the fluent papers of Euler (1707–1783), for example, dealing with quite elementary things, may be read as easily as a detective thriller. A little farther along, a memoir by Lagrange (1736–1813) would make an excellent companion to all the clumsy textbooks of the standard college course in analytical mechanics.

In this connection there is an amusing bit of recent history. At one of the leading American universities the ambitious president had so thoroughly grasped Abel's precept about studying the masters in preference to their pupils, that he proceeded to put it into effect in the freshman class. To aid him in this worthy undertaking, the president called in a specialist in the teaching of science—not a specialist in science. Between them they made up a list of mathematical classics to be read by freshmen in their spare time. These included Newton's *Principia* of 1687 and Einstein's *Theory of General Relativity and Gravitation* of 1916. The last is quite a short trifle. It is the famous paper of which it used to be said that only twelve men in the world could understand it. The president is enthusiastic about the project. The freshmen have not yet been heard from.

THE SPIRIT OF MODERN MATHEMATICS

None of these remarks on the antiquity of the mathematics which passes as sufficient in a liberal education today, or on Abel's sound advice to would-be mathematicians, are intended in any spirit of discouragement. Quite the reverse: by admitting that it is a waste of time for those who are not mathematicians by trade to explore the minutiae of modern mathematics, we shall agree to be content with wider vistas than would satisfy a peering professional. In fact one of the outstanding achievements of the past century was the discovery and exploration of loftier points of view from which many fields of mathematics, both ancient and modern, can be seen as wholes and not as rococo patchworks of dislocated special problems. The details, however, remain matters which only specialists can appreciate.

The summits from which those broader points of view may be gained today seem to us, who did not have the pain of discovering them, to be ridicuously evident. Why were these outstanding peaks not seen before? Viewing the progress of mathematics one might almost be tempted to amend Kant's rhapsody,

> "Two things I contemplate with ceaseless awe,
> The starry Heavens, and man's sense of law."

by striking out "sense of law" and substituting for it "stupidity." The only thing that deters us is the moral certainty that we ourselves are as blind to what stares us in the face as our predecessors were.

If the mathematical spirit of the past hundred years can be described in a phrase, probably *ever greater generality and ever sharper self-criticism* is as just as any. Interest in special or isolated problems steadily diminished as the century advanced, and mathematicians became builders of vast and comprehensive systems of knowledge in which individual theorems were completely subordinated to the grander structure of inclusive theories. The fashioning of ever more powerful weapons for the assault of whole armies of old difficulties, instead of single combat against one at a time, also characterized this golden age of mathematics. And through and over the whole period played an almost continuous brilliance of the most amazing inventiveness the world has ever known.

The other side of the picture is increasing rigor. The so-called obvious was repeatedly scrutinized from every angle and was frequently found to be not obvious but false. "Obvious" is the most dangerous word in mathematics.

CHAPTER II

MATHEMATICAL TRUTH

DESCRIPTIONS OF MATHEMATICS

WHATEVER mathematics was a century ago, it is certainly not today the meagre shadow of itself which the dictionaries make of it. No doubt it takes courage amounting to rashness to quarrel with a standard dictionary, but mathematicians have never been conspicuous for that particular brand of cowardice which submits to the printed word merely because it is fat, black, and backed by authority. Disregarding tradition, some have even framed pithy definitions of their own, intended as improvements on those of the dictionaries.

Unfortunately no two of the definitions are in complete agreement. Each has some high light which reflects the bias of its author, and all taken together might give an impressionistic picture not utterly inadequate. To reproduce all of these attempts to hobble mathematics in a neat phrase would amount to compiling a mathematical dictionary, and the work would be hopelessly out of date long before it was finished. A few examples must suffice.

The first description of mathematics as a whole which need be seriously considered is a much-quoted epigram which Bertrand Russell emitted in 1901.

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

This has four great merits. First, it shocks the self-conceit out of common sense. That is precisely what common sense is for, to be jarred into uncommon sense. One of the chief services which mathematics has rendered the human race in the past century is to put 'common sense' where it belongs, on the topmost shelf next to the dusty canister labled 'discarded nonsense.'

Secondly, Russell's description emphasizes the entirely abstract character of mathematics.

Thirdly, it suggests in a few words one of the major projects of mathematics during the past half century, that of reducing all mathematics and the more mature sciences to postulational form (which will be explained later), so that mathematicians, philosophers, scientists and men of plain common sense can see exactly what it is that each of them imagines he *is* talking about.

Last, Russell's description of mathematics administers a resounding parting salute to the doddering tradition, still respected by the makers of dictionaries, that mathematics is the science of number, quantity, and measurement. These things are an important part of the material to which mathematics has been applied. But they are no more mathematics than are the paints in an artist's tubes the masterpiece he paints. They bear about the same relation to mathematics that oil and ground ochre bear to great art.

Although it is true in a highly important sense, of which examples will appear as we proceed, that we do not know what we are talking about in mathematics, there is another side to the story, which distinguishes mathematics from the elusive reasoning of some philosophers and speculative scientists. Whatever it may be that we *are* talking about in a mathematical argument, we must stick to the subject and avoid slipping new assumptions or slightly changed meanings into the things from which we start.

To be certain that we have not shifted the subject of discussion in an involved and delicate mathematical argument, or to know that our initial assumptions really do contain all that we think we are talking about, is the crux of the whole matter. Time and again mathematicians have been forced to tear down elaborate structures of their own building because, like any other fallible human beings, they have overlooked some trivial defect in the foundations.

Before leaving Russell's definition, let us put two others beside it for comparison. According to Benjamin Pierce (1809–1880), "Mathematics is the science which draws necessary conclusions." As Russell restates the same idea, "Pure mathematics consists entirely of such asseverations as that, if such a proposition is true of *anything*, then such and such another proposition is true. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true." Or again, "Pure Mathematics is the class of all propositions of the form 'p implies q,' where p, q are propositions . . . ."

The evolution of this excessively abstract view of mathematics has been slow, and it is a characteristic product of mathematical activity of the past half century. Not all mathematicians would assent to a definition of this type. Many, particularly those of the older generation, prefer something more concrete.

These estimates may well be enhanced by one from Felix Klein (1849–1929), the leading German mathematician of the last quarter of the Nineteenth Century. "Mathematics in general is fundamentally the science of self-evident things."

This has been reserved for the last because it is so very bad.

In the first place the modern critical movement has taught most mathematicians to be extremely suspicious of "self-evident things." In the second place it is little better than conceited affectation for any mathematician to imply that complicated chains of close reasoning are either easy or avoidable from the beginning. After a problem has had its back broken by half a dozen virile pioneers it is usually simple enough to walk up and dispatch the brute with a single well-aimed bullet. If mathematics is indeed the science of self-evident things, mathematicians are a phenomenally stupid lot to waste the tons of good paper they do in proving the fact. Mathematics is abstract and it is hard, and any assertion that it is simple is true only in a severely technical sense—that of the modern postulational method. The assumptions from which mathematics starts are simple; the rest is not.

Each of the quoted attempts to define mathematics has contributed a valuable touch to the whole picture. These, and the scores of others which have not been mentioned, illustrate the hopelessness of trying to paint a brilliant sunrise in one color. The attempt to compress the free spirit of modern mathematics into an inch in a dictionary is as futile as trying to squeeze an ever-expanding thunder cloud into a pint bottle.

THE POSTULATIONAL METHOD

Less than a century ago it was quite commonly thought that mathematics has a peculiar kind of truth not shared by other human knowledge. For example, Edward Everett in 1870 expressed the popular conception of mathematical truth as follows: "In the pure mathematics we contemplate absolute truths, which existed in the divine mind before the morning stars sang together, and which will continue to exist there, when the last of their radiant host shall have fallen from heaven."

Although it would be easy to match this extravagance by many as wild from recent writings of those who, like Everett, are not mathematicians by profession, it must be stated emphatically that only an inordinately stupid or conceited mathematician would now hold any such inflated estimate of his trade or of the "truths" he manufactures. One very modern instance of the same sort of thing, and we shall pass on to something more profitable. The astronomer and physicist Jeans declared in 1930 that "The Great Architect of the Universe now begins to appear as a pure mathematician." If this high compliment or that of

Everett meant anything, pure mathematicians might indeed feel proud.

Against all the senseless rhetoric that has been wafted like incense before the high altar of "Mathematical Truth," let us put the considered verdict of the man whom most professional mathematicians would agree is the foremost living member of their guild. Mathematics, according to David Hilbert (1862–), is a game played according to certain simple rules with meaningless marks on paper. This is rather a comedown from the architecture of the universe, but it is the final dry flower of a century of progress. The *meaning* of mathematics has nothing to do with the game, and mathematicians pass outside their proper domain when they attempt to give the marks meanings. Without assenting to this drastic deflation of mathematical truth, let us see what brought it about.

The story begins in 1830 with George Peacock (1791–1858) and his study of elementary algebra. Peacock seems to have been one of the first to recognize that algebraical formulas are purely formal—empty of everything but the rules according to which they are combined. The rules in a mathematical game may be any that we please, provided only that they do not lead to flat contradictions like "A is equal to B and A is not equal to

B." The British algebraic school, Peacock, Gregory (1813–1844), Sir William Rowan Hamilton (1805–1865), Augustus De Morgan (1806–1871), and others, stripped elementary algebra of its inherited vagueness and embodied it in the strict form of a set of postulates. As these postulates are illuminating we shall state them in the following chapter in a modern version. Before doing so, however, let us see what postulates are.

A postulate is merely some statement which we agree to accept without asking for proof. A

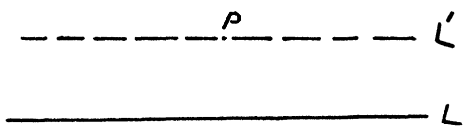

Figure 1

famous example is Euclid's postulate of parallels, one form of which is this: Given a point P in a plane and a straight line L not passing through P, *it is assumed* that precisely one straight line L′ lying in the plane can be drawn through P, such that L and L′ do not meet however far they are drawn." Many geometers after Euclid's time struggled to *prove* that there *is* one such line L′ and, moreover, that there is *only one*. They failed, for the sufficient reason that the postulate is *incapable of proof*.

We return to this in the next chapter. In passing, any modern mathematician will salute Euclid's penetrating genius for recognizing that this complicated statement about parallel lines is indeed a postulate, on a level, so far as Euclid was concerned, with such a simple postulate as "things which are equal to the same thing are equal to one another."

Euclid's postulate illustrates two points about postulates in general. A postulate is not necessarily "self-evident," nor do we ask "is it true?" The postulate is *given; it is to be accepted without argument,* and that is all we can say about the postulate itself. In the older books on geometry, postulates were sometimes called axioms, and it was gratuitously added that "an axiom is a self-evident truth"—which must have puzzled many an intelligent youngster.

Modern mathematics is concerned with playing the game according to the rules; others may inquire into the "truth" of mathematical propositions, provided they think they know what they mean.

The rules of the game are extremely simple. Once and for all the postulates are laid down. These include a statement of all the permissible moves of the "elements"—or "pieces."

It is just like chess. The "elements" in chess are the thirty two chessmen. The postulates of chess are the statements of the moves a player can make, and of what is to happen if certain other things happen. For example, a bishop can move along a diagonal; if one piece is moved to an occupied square, the other piece must be removed from the board, and so on. Only a very original philosopher would dream of asking whether a particular game of chess was "true." The sensible question would be, "Was the game played according to the rules?"

Among the permissible moves of the mathematical game is one which allows us to play. This is the assumption outright that the laws of ordinary logic can be applied to our other postulates. As this blanket postulate is of the highest importance, we shall illustrate its meaning with a simple example.

In the sixth proposition of his first book of *Elements*, Euclid undertakes to prove that *if* the angles ABC and ACB are equal in the triangle as drawn, *then* the side AB is equal to the side AC. His proof is the first recorded example of the indirect method—*reductio ad absurdum* (reduction to the absurd). Euclid provisionally assumes the *falsity* of what he wishes to prove. Namely, he assumes that AB and AC are *unequal*. This leads easily to the conclusion that the angles ABC and ACB are *not* equal. But they were *given* equal. Faced with this contradiction, Euclid concludes by common logic that his provisional assumption that AB and AC are *unequal* must be wrong. Therefore AB and AC must be equal, as this is the only way of avoiding the contradiction.

In this, when fully developed, appeal is made to two of the cardinal principles of Aristotelian logic, the law of contradiction and the law of the excluded

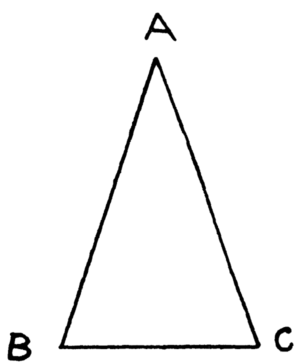

Figure 2

middle. The law of contradiction asserts that no A is not-A; the law of the excluded middle asserts that everything is either A or not-A. Both of these have been accepted until quite recently in all sane reasoning, but both, be it observed, are *postulates*. As we shall see later, the law of the excluded middle has been called into question as a universally valid part of reasoning within the past twenty years by mathematicians. In practically all mathematics of the past century, however, the whole machinery of common logic has been included in the postulates of all mathematical systems. Unless otherwise remarked, this assumption is tacitly made in everything discussed.

Having stated a particular set of postulates, say those of elementary algebra or those of elementary geometry, what next? In the past forty years a beautiful art has developed around postulate systems as things to be studied for their own sake. One question asked about a given set is this. Is the set the most economical? Or is it possible to prune off one and still have a sufficiency? If so, the one that is to be pruned must follow by the rules of logic from the others. With a little practice even amateurs can construct such desirable sets of mutually *independent* postulates. It is at least as amusing a pursuit as solving crossword puzzles or playing solitaire, and it is fully as useful as—whatever anyone cares to mention.

The requirement of independence for our postulate set is not dictated by necessity but by aesthetics. Art is usually considered to be not of the highest quality if the desired object is exhibited in the midst of unnecessary lumber. Many an otherwise good cathedral has been ruined by too many gargoyles.

Are the postulates then completely arbitrary? They are not, and the one stringent condition they must meet has wrecked more than one promising set and the whole edifice reared upon it in the past hundred years. *The postulates must never lead to an inconsistency.* Otherwise they are worthless. If by a rigid application of the laws of logic a set of postulates leads to a contradiction, such as 'A *is* B *and* A *is not* B', the set must either be amended so as to avoid this contradiction (and possibly others), or it must be thrown away. We shall have blundered, and we must start all over again. At this point it is pertinent to ask, *How do we know that a particular set of postulates, say those of elementary algebra, will never lead to a contradiction?*

The answer to this disposes once and for all of the hoary myth of *absolute truth* for the conclusions of pure mathematics. *We do not know, in any single instance, that a particular set of postulates is self-consistent and that it will never lead to a contradiction.* This may seem strong, but the reader will be in a position to judge for himself if he reads the succeeding chapters.

So much for the "absolute truths, which existed in the divine mind before the morning stars sang together"—so far as these were mathematical truths, and so much also for the Greater Architect of the Universe as a pure mathematician. If he he can do no better than some of the postulate systems that pure mathematicians have constructed in the past for their successors to riddle with inconsistencies, the Universe is in a sorry state indeed. The less said about the postulate systems for the universe constructed by scientists, philosophers and theologians the better.

If anyone asks where the postulates come from in the first place, he is harder to answer. Possibly the question is of the kind which mathematicians describe as "improperly posed." Merely because it sounds like a sensible question is no guarantee that it is not as nonsensical as asking when time began.

CHAPTER III

BREAKING BOUNDS

COMMON ALGEBRA

A STATEMENT of what common algebra is from a modern point of view was promised in the preceding chapter. The reader is asked to look rather closely at the simple postulates given, as from them we shall see presently at least one aspect of that process of generalization which was a distinctive feature of much mathematics of the past century.

The letters a, b, c in what follows are to be interpreted as mere *marks* without meaning. Chinese characters or †, *, §, or any other marks would so as well. The signs $\oplus$, $\odot$ may be given any names we please, for example, tzwgb and bgwzt. For the sake of euphony however, they may be read *plus, times.* What follows is a paraphrase of the first part of a paper by E. V. Huntington on *Definitions of a Field by Independent Postulates.* (*Transactions of the American Mathematical Society*, vol. 4, 1903, pp. 27–37). The whole paper is within easy reach of anyone who can read simple formulas.

The underlying idea is that of what we call a *class* in English. We do not define class, but we do assume that, given any class, say C, and an individual, say i, we can recognize intuitively whether i is or is not a member of C. If i is a member of C, we say that *i is in C*. For example, if C is the class of horses, and i is a particular cow, we can point to i and say i is not in C. All this is so simple that the only difficulty is to realize that it is less simple than it seems.

To proceed with common algebra.

We are *given a class* and *two rules of combination*, or *two operations*, that can be performed on any *couple* of things in the class. The operations are written $\oplus$, $\odot$. We *postulate* or *assume* that whenever a and b are in the class, the result, written $a \oplus b$, of operating with $\oplus$ on the *couple* a, b is a unique thing which is in the class. This postulate is expressed by saying that *the class is closed under* $\oplus$. We *postulate* also that the class is closed under the operation $\odot$.

A word as to the reading of formulas. Suppose a and b are in the given class. By our postulate above, $a \oplus b$ is in the class, and therefore it can be combined with any c in the class to give a unique thing again in the class. How shall this last be written? If we get the result from the couple $a \oplus b$, c, we shall write it $(a \oplus b) \oplus c$; if the result is got from the couple c, $a \oplus b$, we shall write it $c \oplus$

$(a \oplus b)$. At this step the hasty may jump to the unjustifiable conclusion that, *necessarily*,

$$(a \oplus b) \oplus c = c \oplus (a \oplus b),$$

where $=$ is the usual sign of equality.

The only things we shall assume about equality are these.

If a is in the class, then $a = a$. This says that a thing "is equal to" itself.

If a, b, c are in the class, and if $a = b$ and $b = c$, then $a = c$. This is Euclid's old friend about things equal to the same thing being equal to one another.

If a, b are in the class, and if $a = b$, then $b = a$.

The postulates proper for common algebra can now be stated in short order. In this particular set there are seven, which we number for future reference.

POSTULATE (1.1) If a, b are in the class, then $a \oplus b = b \oplus a$.

POSTULATE (1.2) If a, b, c are in the class, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

POSTULATE (1.3) If a, b are in the class, then there is an x in the class such that $a + x = b$.

These are merely the familiar properties of algebraic addition precisely and abstractly stated. Subtraction is given by (1.3). Notice that our covering postulate of closure under $\oplus$ permits us to talk sense about $a \oplus b$ and $b \oplus a$ in (1.1), and similarly in the rest. The following three make common multiplication precise. The postulate (2.3) gives algebraic division.

POSTULATE (2.1) If a, b are in the class, then $a \odot b = b \odot a$.

POSTULATE (2.2) If a, b, c are in the class, then $(a \odot b) \odot c = a \odot (b \odot c)$.

POSTULATE (2.3) If a, b are in the class and are *such that* $a \oplus a$ *is not equal to* a, and $b \oplus b$ *is not equal* to b, then there is a y in the class such that $a \odot y = b$.

The seventh and last connects $\oplus$, $\odot$.

POSTULATE 7. If a, b, c are in the class, then $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

Notice that (1.1) and (2.1), also (1.2) and (2.2), differ only in the occurrence of the signs $\oplus$, $\odot$.

If we now replace $\oplus$ by the common $+$, and $\odot$ by $\times$, and then say that the class shall be that of all the numbers, positive, negative, whole or fractional, that ordinary arithmetic deals with, we see that our postulates merely state what every child in the seventh grade knows. *Of course,* to take (1.1), (2.1), we *must* get the same result out of $6 + 8$ as we do out of $8 + 6$, and *of course* 8×6 is the same number as 6×8.

There is no of course about it. Can it be *proved?* Yes, up to a certain extent, provided we *agree to* *stop somewhere and not demand further proof for the things asserted.* This needs elaboration.

In common algebra we point to all the numbers of common arithmetic, as we did just a moment ago, and say *there* is a class, the numbers, and *there* are two operations, common addition and multiplication, which satisfy all our seven postulates.

Examining parts of the curious (2.3), we observe that they amount to forbidding the beginners' sin of attempting to divide by zero.

If then we agree to accept common arithmetic as a self consistent system, we shall have exhibited a consistent interpretation of our seven postulates. Otherwise, granted that arithmetic is self-consistent, we shall have pointed out a self-consistent system satisfying our postulates.

But what about common arithmetic? Why not see what *it* stands on? Do we *know* that the rules of arithmetic *can never* lead to a contradiction? No. In the past half century a host of mathematicians have busied themselves over this. Perhaps the most striking answer is that which bases the numbers on symbolic logic. But on what is symbolic logic based? Why stop there? For the same reason, possibly, that the Hindoo mythologists stopped with a turtle standing on the back of an elephant as the last supporter of the universe. No finality is possible.

Another sort of answer was given by Leopold Kronecker (1823–1891). An arithmetician by taste, Kronecker wished to base all of mathematics on the positive whole numbers 1, 2, 3, 4, ... His creed is summed up in the epigram, "God made the integers, all the rest is the work of man." As he said this in an after dinner speech perhaps he should not be held to it too strictly.

In the paper from which the seven postulates are transcribed it is *proved* that the set is *independent:* no one of the seven can be deduced from the other six.

The system which the seven postulates define is called a *field.* An instance of a field is therefore the common algebra of the schools. The same system, namely a field, can also be defined by other sets of postulates. There is not a unique set of postulates for common algebra, but several, all of which have the same *abstract content.* It is just as if several men of different nationalities were to describe the same scene in their respective languages. The scene would be the same no matter what language was used.

Which of all possible equivalent sets of postulates for a field is the best? The question is not mathematical, as it introduces the elements of taste, or purpose, or value, none of which has yet been given any mathematical meaning. For some purposes a set containing the greatest number of postulates may be preferable. In such a set most, if not all, of the postulates will be simple subject-predicate statements. For other purposes a set in which not all of the postulates are independent might be easier to handle, and so on.

Before leaving this set, let us recall that it contains *all* the rules of the game of common algebra. We can make our moves only in accordance with these rules.

We can make any rules we please to begin with in mathematics, provided they are consistent. But, having made the rules, we must be sportsmen enough to abide by them while playing the game. If the game should prove too hard or uninteresting under the prescribed rules, we are free to make a new set and play accordingly. The exercise of that legitimate license was the source of some of the most interesting mathematics of the golden age.

We have chosen algebra rather than geometry to illustrate postulate systems on account of its greater simplicity. The same sort of thing has been done repeatedly for elementary geometry, for which one of the neatest postulate systems is Hilbert's of 1899–1930.

CHANGING THE RULES

To recall some useful terms, let us name the rule of play given by Postulate (1.1) the *commutative* property of the operation $\oplus$. As the postulate (2.1) says exactly the same thing about $\odot$ that (1.1) does about $\oplus$, we refer to it as the commutative property of $\odot$. Similarly (1.2), (2.2) express the *associative* property, and Postulate 7 is the *distributive* property. These are the familiar names of the schoolbooks on algebra.

The circles in $\oplus$, $\odot$ can now be dropped, as they have sufficiently played their part of emphasizing that we are speaking of whatever satisfies the seven postulates and nothing else. Accordingly we shall now write $a + b$ for $a \oplus b$, and $a . b$ or ab for $a \odot b$, exactly as in any text on algebra.

Suppose now that we rub out one of the postulates, say (1.2), the associative property for addition. Then, whenever $a + (b + c)$ turns up, we can *not* put $(a + b) + c$ for it, as there is no postulate permitting us to do so. We must carry $a + (b + c)$ and $(a + b) + c$ as two distinct pieces of baggage, instead of the one piece we had before. The new algebra is more complicated than the old. Is it any less "true?" Not at all, *provided* we can point to a class of things a, b, c,...... and two operations, our new "plus" and "times," which behave as the *six* postulates we have now laid down require, and which we agree to accept as consistent. Without bothering for the moment whether we can point to an example, let us see how the system defined by the *six* postulates compares with that defined by all *seven* from which it was derived by *suppressing* one *postulate.*

A moment's reflection will show that the new system is *more general,* that is, *less restricted,* than the old. This is plain, because the new system has to satisfy *fewer* conditions than the old, and therefore there is greater freedom within it. Whatever we can say about the new system will hold also for the old. The other way about is false, for *some* things (namely *all* those for which the postulate (1.2) is necessary) can *not* be said about the new.

This illustrates one way of generalizing a mathematical system. We *weaken* the postulates.

More than idle curiosity prompts the next question. By weakening the postulates of a field (common algebra) how many *consistent* systems can be manufactured? I believe the answer has not been printed (it is not mine), but it appears to be 1152. At any rate, the mathematicians of the past century produced well over 200 such systems incidentally in the course of their work on postulates. There are thus 200 or more, possibly 1151, "algebras" in addition to the "common algebra" of the schools, and each of these is *more general* than the common one. The schoolboy of the 22nd Century may have to learn some of these, but he certainly will not be tormented by more than 1152 in all, for that, it can be proved, is the limit of possibilities in this direction.

Anyone except a mathematician may be pardoned for demanding what is the good of this? Isn't the algebra of the high school enough for practical life? A reasonable answer seems to be that high school algebra is either too much or too little for everyday life. Only one person in hundreds ever actually uses the common algebra he learned. But for the many in our technical age who *must* use mathematics in their work far more than common algebra is desirable and often necessary. One example must suffice to give some weight to this assertion.

Open any handbook on mechanics or physics as they are taught in the first two years of college to those who intend to make their livings at applied science, and notice the heavy black letters, usually in Clarendon type, in the formulas. These represent "vectors." A vector is the mathematical name for a segment of a straight line which has both length and direction. A vector **a**, interpreted physically, represents, among other things, a force of stated amount acting in a stated direction. Now follow through a few of the vector formulas. Presently the astonishing fact presents itself that **a** $\times$ **b** is *not* equal to **b** $\times$ **a**, but to *minus* **b** $\times$ **a**.

Vectors are *added* according to our postulates (1.1), (1.2); postulate (2.2) is still good, and postulate (7) is satisfied, all with perfectly sensible physical meanings. But (2.1), the *commutative property* of *multiplication*, has gone overboard, as it *is not true for vectors.* Instead we have **ab** = $-$**ba**. All this, when properly amplified, gives the standard *vector analysis*, without which no one would think nowadays of trying to master mechanics or electricity and magnetism.

Still stranger specimens of our collection have their uses. One, something like vector algebra, discovered by W. K. Clifford in 1872, has just recently proved of great service in studying the complicated mechanics of atoms. Others are of equal interest to mathematicians. Even the freak we suggested by suppressing (1.2) is not without charm.

Another example of generalization (from geometry instead of algebra) will be given presently. For the moment let us glance back. All that has been said is as simple as any interesting game, and is in fact far simpler than chess. Its simplicity has not bloomed over night. Almost a century was required for the perfection of the flower. Sir

William Rowan Hamilton, a universal genius and one of the most creative mathematicians of the golden age, racked his grains for fifteen years in the effort to create a suitable algebra for geometry, mechanics, and other parts of physics. The obstacle which blocked him all those fifteen years was the commutative property of multiplication. Finally the solution flashed on him one day while he was out walking: *throw away the commutative property; a* times *b* is *not* always and everywhere equal to *b* times *a*.

Today a college freshman discards the commutative property without fifteen seconds' thought.

SOURCES OF POSTULATES

With a definite system of postulates now at hand for inspection, we may ask where they came from. To some mathematicians the question is meaningless. Others accept the statement of certain philosophers that the postulates of mathematical systems are derived from experience. This may be satisfactory, provided we know what experience means. But to say that every set of mathematical postulates is a fruit of experience is to stretch the meaning of experience to the breaking point, and to give an answer that is little better than a quibble. If indeed, as Hilbert has asserted, mathematics is a meaningless game played with meaningless marks on paper, the only mathematical experience to which we can refer is the making of marks on paper.

Instead of trying to answer what may be a senseless question by giving a plausible equivocation which any competent mathematician could shoot to pieces in two seconds, let us see how one of the most celebrated systems of postulates actually originated. Anyone who wishes may ascribe the postulates already stated for a field to experience. The set for Lobachewsky's geometry could more properly be credited to a lack of experience in any usual sense of the word.

For centuries before 1826 mathematicians had tried to prove Euclid's postulate of parallels (stated in the preceding chapter) from the remaining postulates of Euclid's geometry. They succeeded in proving that *if* the postulate is so provable, then any one of a large number of equivalent geometrical theorems must be true. Conversely, *if* one of these theorems *is* a consequence of all of Euclid's postulates *except* the one for parallels, then it can be proved that through a point P in a plane can be drawn *exactly one* straight line L′ lying in the plane determined by P and a straight line L not passing through P, such that L and L′ do not meet however far extended.

One of these crucial theorems equivalent to the parallel postulate is this 'obvious' trifle. Given a segment AB of a straight line, (see figure following) and *equal* perpendiculars AC, BD erected at A and B, and on the same side of AB, join CD, and *prove* that *each* of the equal angles (they are easily proved to be equal) ACD, BDC, marked in the figure, *is a right angle:*

Common sense at once "sees" that ACD, BDC are right angles by folding the rectangle over the

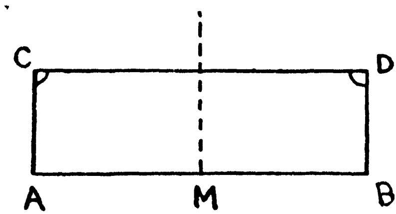

Figure 3

line perpendicular to AB through the middle point M of AB. What common sense thinks it sees is a striking illustration of the fact that mathematics is *not* the science of self-evident things.

Being unable to *prove* that each of ACD, BDC is a right angle by Euclid's geometry *without* using Euclid's parallel postulate, Nicolai Lobachewsky (1793–1856) conceived the brilliant and epoch-making idea of what is equivalent to *postulating* the *assumption* that each angle is *less than* a right angle. With minute care he proceeded to develop the consequences of this hypothesis. It led him to a simple geometry, *just as consistent as Euclid's and equally sufficient for the needs of everyday life*, in which he discovered the following undreamed of situation regarding "parallels."

P is any point not on the straight line L; PH is perpendicular to L; QT and RS are a *particular*

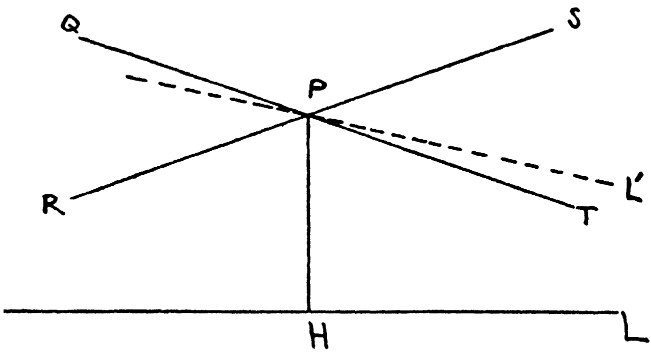

Figure 4

pair of straight lines drawn through P. The angle TPS between RS and QT is *greater than zero;* that is, the lines RS and QT *do not coincide.* Now, in Lobachewsky's geometry, *any line* L' *passing through* P *and* lying within the angle TPS is such that it *never meets* L, however far extended in either direction. So then there are an infinity of "parallels" in Lobachewsky's geometry.

In Euclid's geometry, RS and QT coincide, and there is only one parallel. Lobachewsky calls the two lines PR, PT, neither of which meets L, his parallels, as they both have all the properties of Euclid's one parallel.

Which geometry is "true?" The question is improper; each is self-consistent. And each is sufficient for everyday life.

But why, out of the *three* thinkable possibilities that the equal angles ACD, BDC in the original figure are each *less than*, *equal to*, or *greater than* a right angle, stop with the first two, which give the respective geometries of Lobachewsky and Euclid? There is no compulsion. We may equally well *postulate* that each is *greater than* a right angle. The result is a third geometry, again self-consistent and sufficient for every day life. In this last geometry (developed by Riemann) there are *no* parallels, and a straight line is *closed* and of *finite* length.

Why choose Euclid's in preference to either of the others? Some would say because Euclid's is the simplest of the three to learn, backed as it is by 2200 years of school teaching.

The significant thing for us at present is that Lobachewsky changed the rules of Euclid's game and invented another just as good. This was a tremendous step forward. It showed mathematicians that they might try the same trick of *denying the obvious*, of ignoring or contradicting those things which have been accepted in any region of mathematics, "always, everywhere, and by all," and see what might come out of their boldness.

In geometry alone the outcome during the past century has been sufficiently staggering. Geometries by scores have been created and studied intensively. When first made these were created for their own sake. More than once these manufactured geometries have proved invaluable in science, for which the classical geometry of Euclid is today quite inadequate. We shall return to this later.

Before leaving this, however, let us mention another direction in which geometry freed itself of the shackles of tradition by generalization. Solid space for the Greeks had *three* dimensions, say length, breadth and thickness. When geometry was studied analytically or algebraically instead of synthetically (as was the case up to 1637), the restriction to *three* dimensions no longer was necessary. It was only in the past century however that complete freedom was attained in this direction. First, in analytical mechanics in the Eighteenth Century, it became useful to reason about solid space and time together as a geometry of *four* dimensions. The step from *four* to n (*any* whole, positive, finite number) was taken by Cayley

(1821–1895). From n dimensions to a *countable infinity* of dimensions came considerably later. A countable infinity is as many as there are of *all* the positive whole numbers, 1, 2, 3, ... From geometry of a countable infinity of dimensions to an *uncountable infinity* (as many as there are points on a straight line) of dimensions, was the last step, taken about thirty years ago.

If common sense objects to geometry of four dimensions, it will get little comfort from modern physics. Relativity is based on a particular geometry of four dimensions, and geometry of an *infinity* of dimensions is now commonly used in the mechanics of atoms.

The postulational method of setting up mathematical theories—algebra, geometry, and the rest—was one of the major advances of the past century.

CHAPTER IV

"THE SAME, YET NOT THE SAME"

REALIZATIONS OF COMMON ALGEBRA

No one with a musical ear would mistake a jig for a waltz. The structure of each betrays its nature in the first few bars or phrases. Nor would a musician confuse two waltzes. Although they belong to the same kind of composition, their melodies alone are sufficient to distinguish them immediately.

In mathematics there is frequently discernible a similar structure. Within each of several theories is an inner harmony of pure form, and the form for all is the same. But two theories having the same abstract form may be as different in their outward appearance and in their applications as are two waltzes in sound and emotional appeal. This is not intended as more than a rough description; and the analogy must not be pressed too far.

As a somewhat crude example, let us look first at the postulates of a field stated in the preceding chapter. We shall see that common algebra can be "realized" in any one of at least three ways. In the first the *class* concerned is that of all *rational* numbers; in the second the class is that of all *real* numbers; in the third the class is that of all *complex* numbers. The *structure* of these three fields is the same, namely the postulates (1.1), (1.2), (1.3), (2.1), (2.2), (2.3) and (7). Each is, say, to pursue the analogy, a waltz; the tunes of all three are different. If we work out the consequences of the postulates once for all *abstractly*, without asking for a tune to lighten our labors, we shall have done waltzes completely, all except fitting melodies to particular waltzes. The melodies correspond to the *interpretations* of the things in the given abstract class and those of the abstract operations according to which these things are combined in accordance with the postulates. We use *abstract* to emphasize that *we can say nothing about the system considered*, here a field, *beyond what is explicitly stated in the postulates and what can be deduced by common logic from those postulates alone.* When we say, for example, that the things in the given class are real numbers, we assert something which is not deducible from the postulates, for in them the things were mere marks. By thus putting a definite restriction on the marks, we get a field which is no longer *abstract* or *general,* but *special.* The formulas for this special field will be *instances* of those for the abstract field.

Leaving the analogy, we must first describe what is meant by rational, real and complex numbers.

These notions permeate much of mathematics. It is assumed that we understand what the zero, positive and negative whole numbers, 0, 1, 2, ..., -1, -2 ...... are—a vast assumption in the light of modern critical mathematics.

If a, b are whole numbers, of which b is not zero, the *ratio* of a to b is a/b (the result of dividing a by b). A *rational* number is defined as the ratio of two whole numbers. The class of all whole numbers is a subclass of all rational numbers, as is seen by restricting the divisor b to be 1.

The rational numbers do not include the irrationals. A number is called *irrational* if it is not the ratio of any pair of whole numbers. For example, the square root of 2 is irrational, as can be easily proved by supposing the contrary and getting a contradiction. This fact, by the way, so disconcerted Pythagoras, who had constructed his theory of the universe on the hypothesis that all numbers are rational, that he induced its discoverer to drown himself in order to suppress the awkward theory-destroying fact. So runs the story. It is also reported that the fact had become so notorious in the golden age of Greece that Plato averred that anyone who did not know that the square root of two is irrational (he used different words, suited to geometry), was not a man but a beast.

A part of all the irrationals and all the rationals are swept up into the common class of *real* numbers. To picture these, take any convenient point O on an indefinitely extended straight line, and any convenient length, say an inch, which we agree shall be the unit of measure. Step off 1, 2, 3, ... inches to the right of O, and 1, 2, 3, ... to the left; name the first *positive*, and the second *negative*. The points thus marked, including 0 at O correspond to the whole numbers. Scattered along the line are the points corresponding to the rational

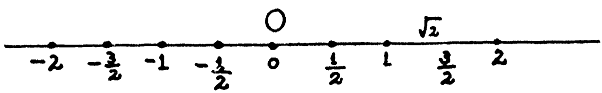

Figure 5

numbers, a few of which are marked in the figure. Where is the square root of 2 on the line? To the right of O and *somewhere between the two rational numbers* 140/100 and 142/100. Being content for the present with that vague somewhere, we remark that *to each point on the line corresponds one and one and only one real number, rational or irrational.* The real numbers are everywhere dense on the line, for between any two we can always locate another —by bisecting the segment joining the two representative points, if no other way suggests itself.

The class of all real numbers is the class whose members correspond, one-to-one, to all the points on the line.

Complex numbers constitute a still vaster assemblage. In describing them, I shall deliberately avoid the perfectly satisfactory way of the high-school texts and return to Gauss. This has two advantages for our purposes. It avoids the legitimate but trivial discussion of what "imaginary" means. "Imaginary" numbers are no more imaginary than are negatives, if we persist in regarding the positive whole numbers as the only true numbers. It also makes it easy to see how mathematicians in the past seventy years *generalized* complex numbers and invented *hypercomplex numbers.*

Following Gauss, we let a, b represent any real numbers, and we create an *ordered couple* (a, b). This ordered couple of real numbers is called a *complex number* if it is made to satisfy certain postulates, of which we shall state only three as samples.

The *sum* $(a, b) \oplus (c, d)$ of the given pair of complex numbers (a, b), (c, d) is defined to be the complex number $(a + c, b + d)$; the result $(a, b) \odot (c, d)$ of *multiplying* the given pair (a, b), (c, d) is defined to be $(ac\text{-}bd, ad + bc)$; equality is defined to mean that $(a, b) = (c, d)$ when and only when

$a = c$ *and* $b = d$. In the above, $a + c$, ac, etc., have their usual meanings as for real numbers in arithmetic.

With these definitions of "addition," "multiplication," and "equality," it is a simple exercise to

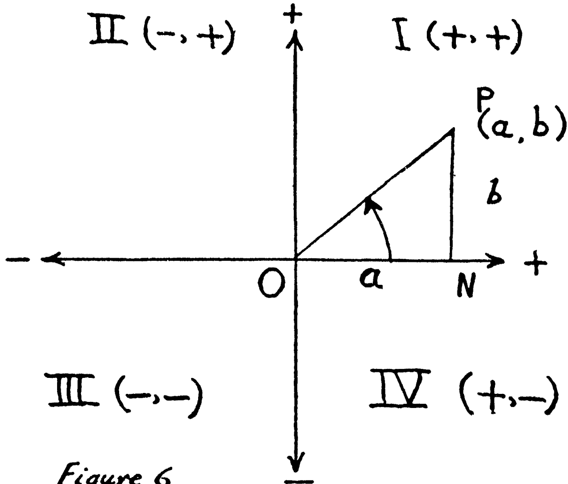

Figure 6

verify that the class of all complex numbers (a, b), (c, d), ... satisfies all the postulates of a field.

In passing, we give the usual geometrical picture of (a, b) (fig 6). Through O draw a perpendicular to the line on which we represented real numbers. Take any point P in the plane fixed by these two lines, and drop a perpendicular PN to the line of real numbers. If the length of ON is measured by the real number a, and that of NP when laid along the line of real numbers is measured by b, we *affix* to the point P the complex number (a, b). If P lies in either of the quadrants labeled I, IV, a is positive; if P lies in II, III, a is negative; if P is in I or II, b is positive; if P is III or IV, b is negative. The pairs $(+, +)$, $(-, +)$, $(-, -)$, $(+, -)$, taken in the order opposite to the motion of the hands of a watch, tell the story on the figure. The "imaginary" square root of minus one has not been mentioned. Whoever cares to look for it will find its image on the vertical line. Notice that (a, b) is also uniquely placed by giving the length of OP and the magnitude of the angle NOP, read as indicated by the arrow. Now OP is a *vector*, whose *magnitude* is the length of OP and whose direction is NOP. This perhaps suggests why complex numbers are of great use in the study of alternating currents, where the vectors concerned are represented graphically.

Out of all this several simple and important things emerge. First, the infinitely rich class of all real numbers is imaged on a mere straight line on the plane picturing the class of all complex numbers, which is infinitely rich in straight lines—they can be drawn in all directions over the whole plane. To anticipate a question that will be discussed later, we state here one of the great discoveries of the golden age. Common sense and all appearances notwithstanding, there are precisely as many real numbers as there are complex numbers. Stated geometrically, this says that on *any* straight line in a plane there are just as many points as there are in the whole plane. If that is not sufficiently jarring to the original sin of our preconceived notions, consider this. In the whole plane there are only as many points as there are on any segment of a straight line, provided only that the segment is indeed a segment and has a length not zero—say a billionth of a billionth of a billionth of an inch. There is a still more striking conclusion of a similar sort. The segment contains as many points as there are in the whole of space of a *countable* infinity of dimensions.

If the reader will look back a few sentences he will see the words "one of the great discoveries of the golden age." That was not a mere rhetorical flourish. It was a historical statement, and was meant to be taken literally. It was neither asserted nor implied that a great discovery is ever necessarily the final one in a given direction. Out of this discovery and what led to it has grown in the past twenty years what is today regarded by many as the turning point in modern mathematics, and we do not yet know whether the signpost reads "Go on" or "Go back."

THE END OF A ROAD

What else do the rational, the real, and the complex numbers give us, beyond a nest of Chinese boxes each of which is enclosed in the one following it? Every schoolboy knows or takes for granted that each of the first two classes of numbers satisfies all the demands of common algebra, and those slightly more advanced know the same for complex numbers. We have then three distinct instances of a field—three waltzes with different melodies. The *structure* of the field is the same in all three, which are *abstractly* identical; the *specialized* fields differ in their interpretations.

Before leaving this, we shall answer the natural question suggested by the rationals, the reals and the complexes. Why not generalize still further, say to triples (a, b, c) of rationals, combined according to appropriate rules?

The answer is again a great landmark. It was proved by Karl Weierstrass (1815–1897) about 1860, and more simply by Hilbert later, that *no further generalization in this particular direction is possible.* We have reached the end of a road. As it is of some importance to understand exactly what Weierstrass proved, I state it more fully. *By retaining* all *the postulates of a field, it is impossible to construct a class of things which satisfies all* *postulates and which is not either the class of all complex numbers or one of the latter's subclasses.*

Here again I have tried to be historically precise. It *was* a landmark of the golden age. In the past six years, however, so broad, so rapid and so deep is the river of mathematical progress, that this landmark has been endangered. Not the fact which Weierstrass and Hilbert thought they proved has yet been swept away, but the type of reasoning which they employed has been called into serious question. Professing no opinion on these matters, which affect all our reasoning in logical patterns inherited from Aristotle, I simply report and pass on.

If complex numbers are the end of this particular road, how shall we progress? Go back and build another! New roads by hundreds were constructed to higher points of view by the mathematicians of the century of progress. One great highway led to the unbounded field of *linear associative algebra*, in which the associative property of multiplication, but not the commutative, is retained in the postulates.

Having acquired from Lobachewsky, Hamilton, and others the habit of denying the obvious, the pioneers might easily have contradicted or denied one or more of the postulates of a field, as we now sometimes do, to reach these vantage points. But this is rather a road for the sophisticated, easy enough to travel after it has been blasted out of the rock and graded. As a matter of fact one of the commanding peaks of the Nineteenth Century, which we could now reach more easily, was discovered otherwise and far from naturally. It all but revealed itself through the mists a score of times to seasoned explorers, who glimpsed its lower slopes but never its summit, for almost a century before a boy of eighteen looked up and saw it all. Less than three years later he was killed in a duel. From the summit which Évariste Galois (1811–1832) discovered, a host of workers, led by Jordan and Kronecker, looked out over the vast domain of algebraic equations and algebraic numbers and perceived order, simplicity, and beauty in what was chaos to the pioneers; another host, led by Felix Klein and ascending yet higher, saw the multitude of geometries which the golden age discovered as isolated provinces united into a single, harmonious pattern of light and shade.

We shall indicate these summits next.

CHAPTER V

OAKS FROM ACORNS

TRANSFORMATIONS

In mathematics it is new ways of looking at old things which seem to be the most prolific sources of far-reaching discoveries. A particular fact may have been known for centuries, and it may have been sterile or of only minor interest all that time, when suddenly some original mind glimpses it from a new angle and perceives the gateway to an empire. What the first flash of intuition sees may take years or even centuries to open up and explore completely, but once a start in the right direction is made, discovery goes forward at an ever increasing speed. Such, in outline, appears to have been the evolution of two of the dominating concepts of the mathematics of the golden age, that of *groups* and *invariants*.

The story begins far back. Distinct traces of the long development are discernible in the work of the Babylonians and the Greeks who, however, never suspected what their regular patterns in tilework and other forms of art meant abstractly, that is, mathematically.

A different approach to the dominating idea seems to have guided the brilliant Arabian algebraists of the Ninth to the Fifteenth Centuries and successive generations of their European followers down to the Eighteenth and first two decades of the Nineteenth Century. But again those who were guided failed to grasp the thread and followed it, if at all, subconsciously.

Regularities and *repetitions* in patterns suggest at once to a modern mathematician the *abstract groups* behind the patterns, and the various *transformations* of one problem, not necessarily mathematical, into another again spell *group* and raise the question *what, if anything, in the problems remains the same*, or invariant, *under all these transformations?* In technical phrase, what are the *invariants* of the *group of transformations?*

When faced with a new problem mathematicians frequently try to restate it so that it is equivalent to one whose solution is already known.

In school algebra, for example, the general equation of the second degree is solved by "completing the square." This reduces the general quadratic to one which we can solve at sight. To recall the steps: we solve $y^2 = k$ for y thus, $y = \pm \sqrt{k}$. We then reduce $ax^2 + 2bx + c = 0$, by completing the square, to

$$\left(x + \frac{b}{a}\right)^2 = \frac{b^2 - ac}{a^2},$$

which is of the *same form* as the easy equation $y^2 = k$. In fact, if we now write $y = x + \frac{b}{a}$, $k = \frac{b^2 - ac}{a^2}$, we have exactly $y^2 = k$. Notice the expression $b^2 - ac$. A remarkable property of this simple expression, considered in a moment, started the whole vast theory of invariance.

Successes such as this were some of the reasons why mathematicians began to study algebraic transformations intensively for their own sake. To illustrate a contributory cause, let us consider two further simple problems, one from elementary algebra, the other from geometry, to see how the comprehensive modern concept of *invariance* originated. Those who have forgotten their first year of school algebra will have to skip the next.

In $ax^2 + 2bxy + cy^2$, express the x, y in terms of new letters X, Y as follows, $x = pX + qY$, $y = rX + sY$. The result is $a(pX + qY)^2 + 2b(pX + qY)(rX + sY) + c(rX + sY)^2$.

Multiply everything out and collect like terms. The result is

$$AX^2 + 2BXY + CY^2,$$

in which A, B, C are the following expressions in terms of a, b, c, p, q, r, s:

$$\begin{aligned} A &= ap^2 + 2bpr + cr^2, \\ B &= apq + b(ps + qr) + crs, \\ C &= aq^2 + 2bqr + cs^2. \end{aligned}$$

We shall leave it to the reader to verify that the new A, B, C and the old a, b, c are connected by the astonishing relation

$$B^2 - AC = (ps - rq)^2 (b^2 - ac).$$

To sum up what has happened, let us write

$$\left.\begin{aligned} x &\rightarrow pX + qY \\ y &\rightarrow rX + sY \end{aligned}\right\},$$
$$ax^2 + 2bxy + cy^2 \rightarrow AX^2 + 2BXY + CY^2,$$
$$B^2 - AC = (ps - rq)^2 (b^2 - ac).$$

The $\rightarrow$ can be read "is transformed into." The indicated transformation of x, y is said to be *linear* (technical term for "of the first degree") in X and Y. The expression $ps - qr$, which depends only on the coefficients p, q, r, s of the transformation of x, y is called the *modulus* of this transformation.

Now look at the summary. It says that $b^2 - ac$ belonging to the original $ax^2 + 2bxy + cy^2$, and $B^2 - AC$, belonging to $AX^2 + 2BXY + CY^2$, differ only by a factor which is the square of the modulus of transformation. For this reason, $b^2 - ac$ is called a *relative invariant* of $ax^2 + 2bxy + cy^2$, "relative," because $b^2 - ac$ is not *absolutely* unchanged under the transformation. If however p, q, r, s are chosen so that $(ps - qr)^2 = 1$, then

$b^2 - ac$ and $B^2 - AC$ are *equal and of the same form*, and we say that $b^2 - ac$ is an *absolute invariant* of $ax^2 + 2bxy + cy^2$ under the given linear transformation. This appears to be the first known instance of such *unchangeableness of algebraic form.*

A mathematician who could look at the relation between $b^2 - ac$ and $B^2 - AC$ and not be at least mildly surprised—provided it was the first time he had seen such a phenomenon—would be little more than an algebraic imbecile. This elementary fact is the acorn, among other things, of the great oak which overshadows modern physics, Einstein's principle of the "covariance of physical laws," and it was planted by Gauss in his immortal *Disquisitiones Arithmeticae* (published in 1801). Cayley, Sylvester, and others made the acorn grow to the oak in 1846–1897.

Our geometrical example requires no algebra. Consider the shadows cast on a wall by a book as it is turned into various positions. The lengths of the sides of the shadow change as the book is moved. What does *not* change? Try it with a flat mesh of straight wires. The *shadow* angles at which the wires intersect and the shadow lengths of the pieces of wire between intersections change in the varying shadows. But an *intersection of two or more wires* remains the same; the shadow wires intersect in the same way as the real wires, and the straight wires *remain straight* in shadow.

The wires represent a simple geometrical configuration of points (intersections) and straight lines. Under the shadow transformation the straightness of the lines is invariant. Further, the intersection of any number of lines is an invariant property, as also is that of the order of any number of intersections lying on one straight line. The shadow is a particular kind of *projection*, like that of a picture on a screen.

Let us recall now that the school geometry (Euclid's) deals almost exclusively with the comparison or measurement of lengths, areas, and angles. For instance, the angle inscribed in a semicircle is a right angle. What becomes of this under projection? It is not invariant, for the circle projects into an ellipse and the right angle loses its "rightness."

Properties of geometrical configurations which are *altered* by projection are called *metric*, since they depend upon measurements. Properties *invariant* under projection are called *projective*. This is merely a description of terms and not an exact or full definition. It is sufficient for our purpose, although in passing it may be mentioned that by taking account of points whose coordinates are complex numbers, the whole of metric geometry can be restated more simply as an episode in projection. The common non-Euclidean geometries also come into the shadow picture.

A PROBLEM IN GEOMETRY

Glancing back at the algebraic example and the geometrical shadows, we see two general problems, one algebraic, the other geometric.

The geometric one is the more easily stated: Given any geometrical configuration, to find all those properties of it which are *invariant* (unchanged) under projection.

This is immediately generalized. Why stop with projection, which is only a particular kind of transformation? We might for instance seek all those properties of extensible, flexible surfaces, like sheets of rubber, which are invariant under stretching and bending without tearing. The geometrical problem now is: Given any geometric thing—configuration, surface, solid, or whatever can be defined geometrically—and given also a set of transformations of that thing or of the space containing it, to find all those properties of the given thing which are invariant under the transformations of the set.

All this can be translated into the perspicuous symbolic languages of algebra and analysis. The last may be very roughly described as that department of mathematics which is concerned with continuous variables. A *variable* is, as its name implies, a mark or letter, say x, which takes on different values successively in the course of a given investigation. For example, the speed of a falling body is not a constant number, say 32 feet per second, but a variable whose numerical value increases continuously from zero (when the body starts to fall) to a greatest speed just as the body strikes the earth.

In passing I must apologize for this very crude description of variables. To state fully what a variable is would take a book. And the outcome would be a feeling of discouragement, for our attempts to understand variables would lead us into the present day morass of doubt concerning the meanings of the fundamental concepts of mathematics. I shall ask the reader to trust his feeling for language and let it go at that: a variable is something which changes. A *continuous real variable* passes through all numbers in a given interval, say from zero to 10, or from zero to infinity.

Now, in 1637 Descartes published his epoch making treatise on analytical geometry. At one step the whole race of mathematicians strode far ahead of the Greek geometers. To understand the connection between the analytical and algebraical aspects of invariance and the geometrical problem of invariance, it is essential to see what Descartes did.

A GLANCE BACK 300 YEARS

In the familiar figure below which every schoolboy (or girl) uses to "graph" one thing or another, is the germ of the revolutionary idea. How could

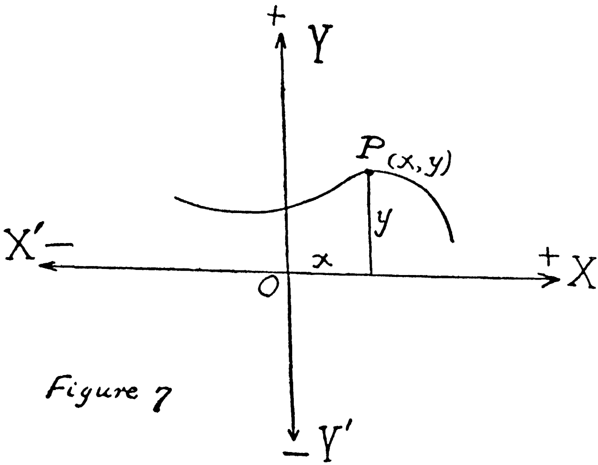

Figure 7

the whole race have missed this till Descartes saw it? A graph is more easily read than any table of numbers.

The point P has the *coordinates* (x, y); x is measured along the X-axis XOX', and y is measured parallel to the Y-axis YOY'. Distances x to the ***right*** of O are *positive*, those to the *left negative;*

distances y measured *up* above XOX' are *positive*, those measured *down* below XOX' are *negative*. The point P is *uniquely* fixed when its coordinates are assigned.

Now consider *all* the pairs (x, y) of coordinates of a point which satisfy a given equation connecting x and y; say the equation is $f(x, y) = 0$ (read as, "equation connecting x and y," or "function of x, y equals zero"). *All* of these pairs will lie on a *certain line*, straight or curved, say briefly on a *curve*, in the plane fixed by XOX' and YOY' and $f(x, y) = 0$ is called *the equation of this curve.* For example, the equation of the circle whose centre is at O and whose radius is 5 is $x^2 + y^2 = 25$.

What Descartes did was this. Instead of studying curves and surfaces by drawing figures as the Greeks had done, he wrote down the equations of the curves considered and proceeded to manipulate these equations algebraically. Then, conversely, he interpreted the resulting algebra in terms of the curves whose equations he had written.

The gain in power was tremendous. A freshman today can prove with ease properties of curves whose difficulties, by the Greek or synthetic method, would have taxed the greatest of the Greeks. This however does not imply that any freshman is necessarily a greater geometer than Euclid or Apollonius, or even Pappus.

The method of Descartes did more. It suggested literally an infinity of interesting curves and surfaces never even imagined by the predecessors of Descartes. Many of these are of the highest importance in practical affairs.

Another great step forward was rendered possible by Descartes' amazingly simple invention of analytical geometry. Many prefer pictures, verbal or graphic, to equations. The invention of analytic geometry enables us to speak the vivid language of geometry about things which are algebraic or analytic.

Last, Descartes potentially freed geometry from the unnecessary restriction to space of three dimensions, although final and complete freedom in this respect was achieved only in the present century. There is no reason why we should suddenly stop with equations in *three* variables. Why not consider any number n? When we do, and use the *language* of geometry, we have a "geometry of n-dimensional space." This also is of great practical use. Thus rigid kinematics is a geometry of 6 dimensions; the theoretical physics of gases is a geometry of 6n dimensions, where n is the number of molecules in the volume of gas considered.

WHAT IS GEOMETRY?

A more appropriate question would be "What *was* geometry in its second golden age of the past century?" In the last ten or twelve years geometry has entered a new phase, vaster and more powerful than ever. The new geometry goes far beyond that which we are about to describe, vast as that was, and it is of unprecedented importance for its suggestiveness in the physical sciences.

The spirit of geometry from at least 1872 to 1922 can not be better or more briefly described than in a famous sentence of Felix Klein. All the astounding inventiveness and infinite variety of geometry during that amazingly prolific half century is seen as one orderly, simple whole from the commanding summit which Klein recognized as the proper point of view to sweep in the whole of the past of geometry and to foresee much of its future. Here is the famous sentence:

"*Given a manifold and a group of transformations of the same, to develop the theory of invariants relating to that group.*"

It is a pity to spoil the beautiful simplicity of this by explanations, but we can be brief. A *manifold* of *n dimensions* is a class of objects which is such that a particular object in the class is completely specified when each of *n* things is given. For instance, a plane is a *two-dimensional manifold of points*, because the plane can be considered as the class of all its points and any point in the plane is completely specified, or uniquely known, when its *two* coördinates x and y are given. Common solid space similarly is a *three-dimensional manifold of points.* I shall leave it to the reader to see that common solid space is also a *four-dimensional manifold of straight lines.* This should rob the "fourth dimension" of some of its silly mystery.

The *transformations* referred to are of the kind which replace each object of the manifold by some definite object of the manifold, or even of another manifold. For instance, we might consider all those transformations of the straight lines of solid space which carry straight lines into other straight lines, or into spheres, for (as the reader may easily think out for himself) common solid space is *four*-dimensional in spheres as well as in lines. (It takes *four* numbers to fix a particular sphere; three to fix the coördinates of the centre, and one to fix the length of the radius). *The number of dimensions of any space depends only upon the elements (points, lines, planes, spheres, circles, etc.) in terms of which the space is described.*

The transformations, according to Klein, must form a group. The postulates for a group are given in the next chapter, and these postulates are the official definition of a group. But as the group is the central and commanding concept of Klein's whole vast program, let us describe its leading property.

Consider a class of things and a set of operations which can be performed on the members of that class. If the result of performing any one of the operations upon any given member (or members) of the class is again a member of the class, we say that the class has the *group property* with respect to the operations. The class then is *closed* under the operations of the set. Thus the class of positive whole numbers 1, 2, 3,...... has the *group property* with respect to addition, for the sum of any two of these numbers is again one of the class. The like holds also for multiplication, but not for subtraction or division.

The *invariants* in Klein's program are those things (properties, actual figures, or what not) that persist, or remain unchanged, under all the transformations, or operations, of a particular given group.

Finally, notice that nothing is said about the number of dimensions of the manifold. This may be 1, or 2, or 3, ... or *n, or it may be infinite.* All possibilities are envisaged in the vast program.

Was Klein's program simply an empty dream, an unnecessary abstraction and generalization of the familiar? Far from it. From that single point of view the geometers of the golden age saw projective geometry, metric geometry of all kinds, Euclid's geometry, innumerable non-Euclidean geometries, geometries of any number of dimensions, and much more, as harmonious parts of Klein's comprehensive, simple program. It was one of the memorable things of all mathematical history, not merely an outstanding achievement of the past century. That the present is going beyond Klein, and ascending higher than he saw, does not diminish the sublimity of his conception.

CHAPTER VI

GROUPS AND MATRICES

AN EXCURSION IN THE PRACTICAL

FROM Klein's program for geometry it is clear that the concept of a *group* dominated at least one major province of mathematics during the past century. Groups also were found to be the structure behind much of modern algebra, in particular the theory of algebraic equations. Wherever groups disclosed themselves, or could be introduced, simplicity and harmony crystallized out of comparative chaos. Finally some modern philosophers became interested in this powerful, unifying mathematical concept of groups as an important phase of scientific thought. As the idea of a group was one of the outstanding additions to the apparatus of scientific thought of the last century, we shall discuss it at some length.

Before proceeding to the official definition of an abstract group, I add a word of caution. Vast as was the panorama swept in from the vantage point of groups, it by no means included the whole of mathematics, either ancient or modern. In many a fertile mathematical province groups either play no part or only a very subordinate one. The whole theory of groups itself is but an incident in the algebra of the past century.

Groups are first subdivided into two grand divisions, *finite* and *infinite*. The number of distinct operations in a finite group is finite; in an infinite group the number of distinct operations is infinite. The subject was developed in the Nineteenth Century by a host of mathematicians, among whom Galois, Cauchy, Jordan, Lie and Sylow may be mentioned.

A *finite* group according to a famous dictum of Cayley in 1854 is defined by its *multiplication table.* Such a table states completely the laws according to which the operations of the group are combined. Here is a specimen which can be easily understood.

	I	*A*	*B*	*C*	*D*	*E*
I	*I*	*A*	*B*	*C*	*D*	*E*
A	*A*	*B*	*I*	*D*	*E*	*C*
B	*B*	*I*	*A*	*E*	*C*	*D*
C	*C*	*E*	*D*	*I*	*B*	*A*
D	*D*	*C*	*E*	*A*	*I*	*B*
E	*E*	*D*	*C*	*B*	*A*	*I*

This group contains the six operations I, A, B, C, D, E. We shall state what the table says about any pair of these operations, say B and D. Take any letter, say B, from the lefthand vertical column, and any letter, say D, from the top horizontal row, and see the entry C in the table where the *B-row* and the *D-column* intersect. It is just as if we were to *multiply* B by D, say $B \times D$, and get the answer C. Instead of writing $B \times D$, we shall write BD, which says to take B from the *left*, D from the *top*, and find where the corresponding row and column intersect. This gives the result C; so we write $BD = C$.

What about DB, found according to the same rule? It is not equal to C, but to E; namely, $DB = E$. So in this kind of *composition*, BD and DB are not necessarily equal. The reader may easily satisfy himself that although the commutative law has gone, the associative is still valid. For instance $(AB)\,C = A\,(BC)$.

Let now x be *any* member of a given class on which I, A, B, C, D, E operate. We *postulate* that the result of operating with any one of I, A, B, C, D, E on x *gives another member of the class*. Let us write $B\,(x)$ (read, "B on x") for the *result* of operating on x with B. By our postulate this is some member of the given class, so we can operate on $B\,(x)$ with D. The result is written $BD\,(x)$, which is again in the class. Now, the assertion of the table that $BD = C$ says that, instead of performing the operations B, D *successively*, first B and then D, we could reach the same final result in *one* step, by performing the operation C on x. Thus, *the class is closed with respect to the operations* I, A, B, C, D, E. For the results of performing the operations of the set successively are always in the set. If the reader does not believe this, let him follow the rule which gives $BD = C$, $DB = E$, $CE = A$, $EC = B$, etc., and try to escape from the table. Lay aside this book for a moment and reflect on the miracle that such closed, finite sets actually exist.

Notice the effect of operating with I. The table says that $AI = IA = A$; $BI = IB = B$, and so for all. Thus I as an operation changes nothing; it is called the *identity*.

The last thing to be observed attentively is this. Given *any one* of I, A, B, C, D, E, say X, there is *always exactly one other of* the six, say Y such that $XY = I$. Further, for every such pair, X, Y it is true that $XY = YX$. It is not asserted that X, Y are *necessarily* distinct. For example, if X is the particular operation B, then the table says that $Y = A$, because $BA = I$; if X is E, then Y also is E. Two operations X, Y such that $XY =$ the identity are called *inverses* of one another. The table states that *each element of the set has a unique inverse.*

A set of operations having all of the foregoing properties is called a *group.* The definition by postulates will be given presently.

For the moment let us see that an instance of the group defined by the specimen multiplication table actually exists. There are dozens of them—all in different parts of mathematics. Here is a very simple specimen from arithmetic. Start with any number different from zero, say x. We can *subtract x from* 1, and we can *divide* 1 *by* x, getting the new numbers $1 - x$ and $1/x$. Repeat these operations on the new numbers. Then $1 - x$ gives back x and a new number $1/(1 - x)$; $1/x$ gives the new number $1 - 1/x$ or $(x - 1)/x$, and gives back x. Keep this up forever. You can never get but one or other of the six numbers $x, 1/x, 1 - x, 1/(1 - x)$, $(x - 1)/x, x/(x - 1)$. Now let I be the operation which transforms x into itself, $I(x) = x$; let B be the operation which transforms x into $(x - 1)/x$, or $B(x) = (x - 1)/x$; and so on, with $C(x) = 1/x$, $D(x) = 1 - x$, $E(x) = x/(x - 1)$. A little patience will show that *these* I, A, B, C, D, E satisfy the multiplication table.

The number of different operations in a group is called its *order.* Thus our group is of order 6. Looking at the table more closely, we see several smaller groups within the whole group, for example those whose multiplication tables are

$$\begin{array}{c|c} & I \\ \hline I & I \end{array}, \qquad \begin{array}{c|cc} & I & C \\ \hline I & I & C \\ C & C & I \end{array}, \qquad \begin{array}{c|ccc} & I & A & B \\ \hline I & I & A & B \\ A & A & B & I \\ B & B & I & A \end{array}$$

whose respective orders are 1, 2, 3. Now 1, 2, 3 are *divisors* of 6, and we have illustrated a fundamental theorem of groups, the *order of any subgroup of a given group is a divisor of the order of the group.*

The following postulates for a group should now be intelligible.

We consider a class and a rule, written as $\circ$, by which the two things A, B in any ordered couple (A, B) of things in the class can be combined so as to yield a unique thing which is again in the class.

The result of combining A, B in the ordered couple (A, B) where A and B are any things in the class, is written $A \circ B$.

POSTULATE. (*Closure under* $\circ$). If A, B are in the class, then $A \circ B$ is in the class.

POSTULATE. (*Associativity of* $\circ$). If A, B, C are in the class, then $(A \circ B) \circ C = A \circ (B \circ C)$.

POSTULATE. (*Inclusion of identity*). There is a unique thing I in the class such that $A \circ I = I \circ A$ for every thing A in the class.

POSTULATE. (*Unique inverse*). If A is any thing in the class, there is a unique thing, say A', in the class such that $A \circ A' = I$.

The foregoing postulates define a *group:* the *class* is said to be *a group under* (or with respect to) *the composition* $\circ$. The postulates contain redundancies, but are more easily seen in the above inelegant form. The A, B, C, ... are our previous "operations."

It is instructive to compare the postulates for a group with those for a field. It will be seen that, if we *suppress the commutative property* of multiplication in a field, the remaining postulates for multiplication are those of a group, and likewise for addition.

If the composition $\circ$ does have the commutative property (as in the arithmetical examples above), the group is called *commutative*, or *Abelian* (after Abel).

Let us glance back here at the linear transformation

$$\left.\begin{aligned} x &= pX + qY \\ y &= rX + sY \end{aligned}\right\}$$

of Chapter V. Merely write down the coefficients p, q, r, s of this transformation, as Cayley did in a fundamental "Memoir on Matrices" in 1858, thus

$$\left\| \begin{matrix} p, & q \\ r, & s \end{matrix} \right\|.$$

This is called a *matrix* (of *order 2*, since there are 2 rows and two columns). What of it? the skeptical reader may ask. I refer him to the physicists for the moment, until an answer can be indicated shortly. At present I wish merely to emphasize that matrices were invented in 1858 by Cayley for the purposes of pure mathematics, and neither he nor anyone else dreamed that 88 years later they would prove to be a subtle clue to some of the deepest mysteries of the physical universe.

Cayley dealt directly with the matrix instead of with the linear transformation of which it is the skeleton. An important thing about matrices is the way they are *combined*, or *operated upon*, or, in technical phrase, *multiplied*. The rule is illustrated thus

$$\left\| \begin{matrix} p, & q \\ r, & s \end{matrix} \right\| \times \left\| \begin{matrix} P, & Q \\ R, & S \end{matrix} \right\| = \left\| \begin{matrix} pP + qR, & pQ + qS \\ rP + sR, & rQ + sS \end{matrix} \right\|,$$

where $\times$ is read "times," and the matrix on the right of the sign $=$ is *by definition* the result of performing $\times$ on the given matrices in the given order. As numerical examples:

$$\begin{Vmatrix} 1, & 3 \\ 2, & 4 \end{Vmatrix} \times \begin{Vmatrix} 3, & 5 \\ 6, & 7 \end{Vmatrix} = \begin{Vmatrix} 21, & 26 \\ 30, & 38 \end{Vmatrix};$$

$$\begin{Vmatrix} 3, & 5 \\ 6, & 7 \end{Vmatrix} \times \begin{Vmatrix} 1, & 3 \\ 2, & 4 \end{Vmatrix} = \begin{Vmatrix} 13, & 29 \\ 20, & 46 \end{Vmatrix}.$$

The multiplication of matrices is *not commutative*, as shown by this example or the rule.

Suppose now that we perform two linear transformations in succession. The matrix of the single resulting linear transformation is obtained by the rule above. The extension to matrices of any order is immediate. From this remark it may be surmized that groups and matrices are intimately connected, and this is the fact. Cayley and his successors perfected the theory of matrices; the theory of groups is a mine for our successors to exhaust.

Let us glance back. No man, I believe, no matter how practical, could point to a more conspicuous example than matrices of the apparently futile things over which mathematicians labor as few others ever dream of laboring. And it would be difficult to find a better instance of the historical fact that the significant advances of mathematics and not a few of those of science are inspired by the spirit of pure mathematics. Just as "beauty is its own excuse for being," so mathematics needs no apology for existing. I apologize, on the contrary, for pointing out presently that matrices do have a non-mathematical use, and a highly important one at that. This use, by the way, is but one of many.

After such an instance as this, scientists, educators, and others may be more willing to let mathematics develop according to its own nature, instead of insisting, as they have sometimes done, that it should draw its life from finance, bridge-building, statistics, or whatever happens to be popular at the moment in physics. As remarked before, the deliberate effort to follow immediate utility in mathematics almost invariably leads to second or third rate work, and more often than not the very utility which is narrowly sought turns out to be not so great after all.

The mathematics of what many mean by everyday life,—practical mechanics, buying and selling, and the other necessary activities by which we live more or less from hand to mouth, is for the most part worked out and reduced to simple rules of thumb. But that which is of vital importance in modern life, which is based on an ever expanding science and an ever more scientific technology, is not simple, and it is not a matter for the engineering handbooks. It is partly in process of creation. Mathematicians may safely be left to follow their own bent as their contribution to this age of science. What they did in the past century is enough for a vast region of science and technology as they exist today; what mathematicians as professionals are interested in today will, if there is any continuity at all in scientific and industrial history, be the indispensable framework of the science and technology of tomorrow.

To return to matrices, one of the fair examples of the foresight of mathematics. When Heisenberg in 1926 was casting about for some adequate means of formulating the mechanics of the atom—possibly the dominant interest in Twentieth Century physics—he found exactly what was required in the theory of matrices, with its queer "multiplication" in which *a* times *b* is not necessarily *b* times *a*. From that work developed with amazing speed the new mechanics and the new physics of spectra and atoms.

In 1926, and again in 1931, Hermann Weyl wrought the new physics into a beautiful, suggestive pattern in which the theme is the theory of groups and the interpretation of quantum mechanics in terms of that great, abstract theory. In this interpretation some of the most abstract and advanced labors of algebraists for the past 70 years are drawn on and pressed into service to the limit. The advanced analysis, integral equations, and the rest, of the past thirty years is also utilized to the full.

INFINITE GROUPS

A word must be said about infinite groups. These again fall into two grand divisions. In the first, the distinct things are *denumerable*, that is, the things in the group can be counted off 1, 2, 3, ..., but *we never come to the end.* Such groups are infinite and *discrete.* In *continuous* groups the number of distinct things is infinite, but not denumerable; the things can not be counted off 1, 2, 3, ..., but are as numerous as the points on a line.

Continuous groups arise in the following way among others. In school geometry it is *assumed* that a plane figure, say a triangle, can be moved about all over the plane and retain its shape (size of angles and length of sides). Consider the group of all motions of a rigid figure in a plane. Evidently the group contains *infinitesimal* transformations, for we can shift the figure from one position to another by stages as small as we please.

Another example of a group consisting of infinitesimal transformations is that of the rotations of a rigid, solid body about a fixed axis. Either the body as a whole may be thought of as being moved from one position to another, or the motion may be realized by subjecting each individual point to an appropriate transformation. Both points of view are useful.

Now let us recall that the equations of mechanics and those of classical mathematical physics are *differential equations*. Roughly, such equations express laws concerning rates of change of one or more continuously varying magnitudes with respect to one or more others. As a simple example, the velocity of a falling body is the "rate" of change of position with respect to time. The vast theory of differential equations was greatly furthered by the introduction of continuous groups into its study. For instance, the central equations of higher dynamics, those named after their discoverer, Hamilton, when viewed from the standpoint of continuous groups become much clearer than before.

The study of such groups absorbed the working lives of many mathematicians from 1873 to the early years of this century, when interest diminished, owing to a great memoir published in 1894 by Élie Cartan, which disposed of several of the main problems. The theory of continuous groups in its broad aspects was almost exclusively the work of one man, Sophus Lie (pronounced Lee), 1842–

1899. But with the new physics, beginning with general relativity in 1915, and continuing with the quantum mechanics of 1926– (?), continuous groups suddenly were seen to be of fundamental importance in the description of nature. New and yet more general geometries, suggested partly by physics, are being created in swarms, and in this outburst the theory of continuous groups has been at least a highly suggestive guide. In this latest renaissance of infinitesimal geometry, Cartan has been and is a leader.

The foregoing is mentioned to introduce another landmark of progress. In the past seven years Klein's great program, which directed geometry for half a century, has been found insufficient. Geometry is blooming again, more freely and more luxuriantly than ever, uncramped by the limitations imposed by the theory of groups. The new geometries—of the highest suggestiveness for physical science—do not conform to the pattern of the group. Galileo was right. The world does move.

THE ICOSAHEDRON

Although it is not our intention to discuss special results, we may close this description of groups by referring to one which would have delighted Pythagoras and have caused him to sacrifice at least a thousand oxen to his immortal gods. The story covers nearly 2200 years. Only the high points can be indicated.

The Greek geometers early discovered the 5 regular solids of Euclidean space,—the tetrahedron, cube, octohedron, dodecahedron, and icosahedron of 4, 6, 8, 12 and 20 sides respectively, and proved that there are no others. This discovery begot much of the incredible mysticism of later and less exact thinkers.

Our next high point is about 2000 years farther on. For over two centuries algebraists had tried in vain to solve the general equation of the fifth degree,

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$$

until Abel in 1826 and Galois in 1831 *proved* that it is impossible to express x by any combination of the given numbers a, b, c, d, e, f, using only a *finite* number of additions, multiplications, subtractions, divisions and extractions of roots. Thus it is *impossible to solve the general equation of the fifth degree algebraically*. On the eve of that stupid duel in which he was killed, Galois, then in his 21st year, wrote out his mathematical testament, in which, among other tremendous things, he sketched a great theorem concerning all algebraic equations. He reduced the problem of the *alge-*

braic solution of equations to an equivalent, approachable one in groups. As this is an outstanding landmark in algebra, I shall state Galois' theorem, in the hope that some may be induced to go farther and find out for themselves exactly what it means: *an algebraic equation is algebraically solvable, if, and only if, its group is solvable.* No more technical knowledge is necessary to follow the proof than is possessed by high school graduates. As a consequence of this perfect theorem, it is *impossible to solve the general equation of any degree greater than 4 algebraically.*

In 1858 Hermite *solved the general equation of the fifth degree*, not algebraically, for that would have been to do the impossible, which is too much even for mathematicians, but by expressing x in terms of elliptic modular functions (a sort of higher species of the familiar trigonometric functions).

Our last peak was discovered by Klein, who showed in 1884 that the profound work of Hermite was all implicit in the properties of the group of rotations about axes of symmetry which change an icosahedron into itself—that is, which twirl the solid about so that, say, a given vertex slips over to the place where some other vertex was, and so for all in every rotation. There are 60 such rotations.

That the rotations of an icosahedron and the general equation of the fifth degree should be unified from the higher standpoint of groups, is a good illustration of the power of the concept of an abstract group

The far reaching power of the theory of groups resides in its revelation of identity behind apparent dissimilarity. Two theories built on the same group are structurally identical. The more familiar is worked out; the results are then interpreted in terms of the less familiar.

CHAPTER VII

THE QUEEN OF MATHEMATICS

AN UNRULY DOMAIN

GAUSS crowned arithmetic the queen of mathematics. Gauss lived from 1777 to 1855, and to his profound inventiveness is due more than one strong river of mathematical progress during the past century. He also made outstanding contributions to the science of his time, notably to electromagnetism and astronomy. His opinions therefore carry weight with all mathematicians and with some scientists.

Arithmetic to Gauss, as to the Greeks, was primarily the study of the properties of whole numbers. The Greeks, it may be remembered, used a different word for calculation and its applications to trade. For this practical kind of arithmetic the aristocratic, slave-owning Greeks seem to have had a sort of contempt. They called it logistica, a name which survives in the logistics of one modern school in the logic and foundations of mathematics.

In arithmetic as in all fields of mathematics during the past century discovery went wide and far. But there was one most significant difference between this advance and the others. Geometry, analysis, and algebra each acquired one or more vantage points from which to survey its whole domain; arithmetic did not.

The Greeks left no problem in geometry which the moderns have failed to dispose of. Faced by some of the trifles which the Greeks left in arithmetic we are still baffled. For instance, give a rule for finding all those numbers which, like 6, are the sums of all their divisors less than themselves, $6 = 1 + 2 + 3$, and prove or disprove that no odd number has this property. To say that arithmetic is mistress of its own domain when it cannot subdue a childish thing like this is undeserved flattery.

The theory of numbers is the last great uncivilized continent of mathematics. It is split up into innumerable countries, fertile enough in themselves, but all more or less indifferent to one another's welfare and without a vestige of a central, intelligent government. If any young Alexander is weeping for a new world to conquer, it lies before him.

Lest this estimate seem unduly pessimistic, let us not forget that in each of the several countries of arithmetic there was considerable progress in the past century. Indeed, two or three of the splendid things done are comparable to anything in geometry, with this qualification, however: no one advance affected the whole course of development. This possibly is due to the very nature of the subject.

Among the notable advances is that which revealed one source of some of those mysterious harmonies which Gauss admired in the properties of whole numbers. This was the creation by Kummer, Dedekind and Kronecker of the theory of algebraic numbers. In this particular field the invention of ideal numbers is comparable to that of non-Euclidean geometry. Another striking advance was the brilliant development of the analytic theory of numbers during the past thirty years. Of isolated problems inherited from the past that have been successfully grappled with we may mention in particular Waring's of the Eighteenth Century. Another result of singular interest was the proof that certain numbers are transcendental, and the construction of many such numbers. We shall briefly indicate the nature of all these things presently. These preliminaries may well be closed with the following quotation.

"The higher arithmetic," wrote Gauss in 1849, "presents us with an inexhaustible storehouse of interesting truths—of truths, too, which are not isolated, but stand in the closest relation to one another, and between which, with each successive advance of the science, we continuously discover new and wholly unexpected points of contact. A great part of the theories of arithmetic derive an additional charm from the peculiarity that we easily arrive by induction at important propositions, which have the stamp of simplicity upon them, but the demonstration of which lies so deep as not to be discovered until after many fruitless efforts; and even then it is obtained by some tedious and artificial process, while the simpler methods of proof long remain hidden from us."

ALGEBRAIC NUMBERS

The positive, zero and negative whole numbers of common arithmetic are called *rational* integers, to distinguish them from *algebraic integers*, which are defined as follows.

Let $a_0, a_1, a_2, \ldots\ldots, a_{n-1}, a_n$ be $n + 1$ given rational integers, of which a_0 is not zero, and not all of which have a common divisor greater than 1. It is known from the fundamental theorem of algebra (first proved in 1799 by Gauss) that the equation

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$$

has exactly n roots. That is, there are exactly n real or complex numbers, say $x_1, x_2, \ldots\ldots, x_n$, such that if any one of these be put for x in the equation, the left hand side becomes zero. Notice that no kind of number beyond the complex has to be created to solve the equation. If $n = 2$, we have the familiar fact that a quadratic equation has precisely two roots. For emphasis I repeat that $a_0, a_1, a_2, \ldots\ldots, a_n$ in the present discussion are rational integers, and that a_0 is not zero. The n roots $x_1, x_2, \ldots\ldots, x_n$ are called *algebraic numbers. If a_0 is* 1, these algebraic numbers are called *algebraic integers*, which are a *generalization* of the rational integers. For instance, the two roots of $3x^2 + 5x + 7 = 0$ are algebraic numbers; the two roots of $x^2 + 5x + 7 = 0$ are algebraic integers.

A rational integer, say n, is also an algebraic integer, for it is the root of $x - n = 0$, and so satisfies the general definition. But an algebraic integer is not necessarily rational. For instance, neither of the roots of $x^2 + x + 5 = 0$ is a rational number, although both are algebraic integers. In the study of algebraic numbers and integers we have another instance of the tendency to generalization which distinguishes modern mathematics.

Omitting technical details and refinements, we shall give some idea of a radical distinction between rational integers and those algebraic integers which are not rational. First we must state what a field of algebraic numbers is.

If the left hand side of the given equation $a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n = 0$, in which $a_0, a_1, \ldots\ldots,$

a_n are rational integers, can *not* be split into two factors each of which has again rational integers as coefficients, the equation is called *irreducible* of *degree n*.

Now consider all the expressions which can be made by starting with a particular root of an irreducible equation of degree n (as above) and operating on that root by addition, multiplication, subtraction and division (division by zero excluded). Say the root chosen is r; as specimens of the results we get $r + r$, or $2r$, r/r or 1, $r \times r$ or r^2, then $2r^2$, and so on indefinitely. The set of all such expressions is evidently a field, according to our previous definitions; it is called the *algebraic number field* of *degree n generated by r.* This field will contain algebraic numbers and algebraic integers. It is these integers at which we must look, after a slight digression on rational integers.

The rational primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ..., namely the numbers greater than 1 which have only 1 and themselves as divisors. The *fundamental theorem of arithmetic* states that a rational integer greater than 1 is either a prime or can be built up by multiplying primes in essentially *one way only.* For instance, $100 = 2 \times 2 \times 5 \times 5$, $105 = 3 \times 5 \times 7$. This is so well known that some writers of school books assert it to be "self-evident," which is another instance of the danger of the obvious in mathematics. Euclid gave a beautiful proof of this theorem, one of the gems of all his work. If the reader has never seen a proof, it may puzzle him to make one.

Primes in algebraic numbers are defined exactly as in common arithmetic. But the "self-evident" theorem that every integer in an algebraic number field can be built up in essentially one way only by multiplying primes is, unfortunately, *false*. The foundation has vanished and the whole superstructure has gone to smash.

One should not feel unduly humble at jumping to this particular obvious but wrong conclusion. More than one first rank mathematician less than a century ago did the same. One of them was Cauchy, but he soon pulled himself up short. In *some* algebraic number fields an algebraic integer can be built up in more than one way by multiplying primes together. This is chaos, and the way back to order demanded high genius for its discovery.

The way in which the whole question originated was this. Fermat (1601–1665) bequeathed this teaser to his exasperated successors. "It is impossible to find three rational integers x, y, z all different from zero, and a rational integer n greater than 2, such that $x^n + y^n = z^n$. The exception 2 is necessary; for instance, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$.

The mathematician and business man Fermat was well known for his probity. He asserted that he had an "extremely simple proof," which the margin of his book was too narrow to contain. For nearly 300 years arithmeticians and others have broken their heads over Fermat's assertion and so far haven't made a dent in it. The assertion remains unproved, although it is known to be true for numerous values of n. A *general* proof is what is wanted.

About 1845 E. E. Kummer (1810–1893) thought he had one. His friend Dirichlet pointed out the mistake. Kummer had assumed the truth of that obvious but not always true theorem about the prime factors of algebraic integers. He set to work to restore order to the chaos in which arithmetic found itself, and in 1847 published his *restoration* of the fundamental law of arithmetic for the particular fields connected with Fermat's assertion. This achievement is usually rated as of greater mathematical importance than would be a proof of Fermat's theorem. To restore unique factorization into primes in his fields, Kummer created a totally new species of number, which he called *ideal*.

In 1871 Richard Dedekind (1831–1916) did the like by a simpler method which is applicable to the integers of *any* algebraic number field. Rational arithmetic was thereby truly generalized, for the *rational* integers are the algebraic integers in the field generated by 1 (according to our previous definitions).

Dedekind's "ideals," which replace numbers, stand out as one of the memorable landmarks of the past century. I can recall no instance in mathematics where such intense penetration was necessary to see the underlying, true pattern beneath the apparent complexity and chaos of the facts, and where the thing seen was of such shining simplicity. A rough idea of Dedekind's "ideals" can be glimpsed from their very degenerate form for the rational integers.

Consider the fact that 3 divides 12 arithmetically. The quotient is 4. Therefore 12 is four times as great as three. The last is precisely what we *must not look at,* obvious and true as it is. On the contrary we need to see that 4, from another point of view, is really bigger than 12, in the sense of greater inclusiveness. Precisely: we no longer look at 3 and 12, but at the respective *classes* of rational integers which we get when we multiply each of 3, 12 by *all* the rational integers in turn. Thus, some of the integers in the class so generated by 3 are -9, -6, -3, 0, 3, 6, 9, ..., and similarly from 12 we have the specimens -36, -24, -12, 0, 12, 24, 36, ... The *class* generated by 3 is called the principal *ideal* 3, and similarly for 12. The ideal 3 *contains*, or *includes* the ideal 12. That is, every rational integer in the ideal 12 is in the ideal 3. The other way about is false; for instance, the ideal 12 does not contain 9.

The reader may easily see that all the properties of common arithmetical division persist if we make the following changes: replace every rational integer by the principal ideal which it generates, and replace the word "divides" by "contains."

It was a natural extension of this inverted way of looking at division which restored the fundamental theorem of rational arithmetic to the vaster domain of algebraic number fields. The extension deals with classes defined, not by a single integer, but by a set of n integers.

As might be expected from the definition of algebraic numbers as roots of algebraic equations, the theory of groups plays an important part in algebraic number fields. Here also the marvellous creations of Galois in the theory of algebraic equations have free scope. Exploration in this territory is still in progress, and much is being discovered.

TRANSCENDENTAL NUMBERS

A mere glance at these must suffice. A number which is *not algebraic* is called *transcendental.* Otherwise stated, a transcendental number satisfies no algebraic equation whose coefficients are rational numbers. It was only in 1844 that the existence of transcendentals was proved by Joseph Liouville (1809–1882). The transcendental numbers, hard as they are to find individually, are *infinitely more numerous* than the algebraic numbers. A very famous transcendental is π (pi), the ratio of the circumference of a circle to its diameter. To 7 decimals, $\pi = 3.1415926\ldots\ldots$, and it has been somewhat uselessly computed to over 700. In 1882 Lindemann, using a method devised in 1873 by Hermite, proved that π is transcendental, thus destroying for ever the last slim hope of those who would square the circle—although many of them don't know even yet that the ancient Hebrew value 3 of π was knocked from under them centuries ago.

In 1900 Hilbert emphasized what was then an outstanding problem, to prove or disprove that $2^{\sqrt{2}}$ is transcendental. The rapidity of modern progress can be judged from the fact that Kusmin in 1930 proved a whole infinity of numbers, one of which is Hilbert's, to be transcendental. The proof is quite simple.

WARING'S CONJECTURE

Fermat proved that every rational integer is a sum of *four* rational integer squares. Thus 10 =

$0^2 + 0^2 + 1^2 + 3^2$, $293 = 2^2 + 8^2 + 9^2 + 12^2$, etc. In 1770 E. Waring guessed that every rational integer is the sum of a *fixed* number N of n th powers of rational integers, where n is any given integer and N depends only upon n. For $n = 3$, the required N is 9; for $n = 4$, it is known that N is not greater than 21. Hilbert, by most ingenious reasoning, proved Waring's conjeccture to be correct in 1909. In 1919 G. H. Hardy (1878–) applying the powerful machinery of modern analysis, gave a deeper proof, the spirit of which is applicable to many other extremely difficult questions in arithmetic. This advance was highly significant for its joining of two widely separated fields of mathematics, *analysis*, which deals with the *uncountable*, or *continuous*, and *arithmetic*, which deals with the *countable*, or *discrete*.

Finally, quite recently, Winogradov has brought some of these difficult matters within the scope of comparatively elementary methods. Here again progress is increasingly rapid. The conquests being made today in this field would have seemed to the men of a hundred years ago to be centuries beyond them.

ANALYTICAL ARITHMETIC

In our generation we have seen the application of analysis to arithmetic on a scale which only fifty years ago was undreamed of. As one type of problem in this province, we may cite the distribution of primes. The question is to state how many primes there are below a given limit, say a billion billion. To find and count them is humanly impossible. The problem as stated seems to be hopeless; an *exact*, terminated formula in terms of simple expressions is out of the question. But we can ask for a formula of this kind: if $P(x)$ denotes the number of primes not greater than x, to find an expression containing x such that $P(x)$ divided by this expression tends to the limiting value 1 as x tends to infinity. This has been solved; the required expression is x divided by the logarithm of x. The solution of this age-old problem was given in 1896 by J. Hadamard and de la Vallée Poussin simultaneously and independently. Subsequent work deals, roughly, with estimating the error committed in distribution formulas by stopping short of the end. A little more precisely, the analytic theory of numbers is largely concerned with determining the *order* (relative size) of the errors made if we take an *approximate* enumeration in a particular problem concerning a class of numbers instead of the exact enumeration. In this the leaders are G. H. Hardy, Edmund Landau, and J. E. Littlewood.

The broader significance of all this work is its fusion of modern analysis and arithmetic into a a powerful method of research in the theory of numbers. Fifty years ago such a union would have been an idle dream.

One further problem may close this sketch. In 1742 Chr. Goldbach, on only scanty empirical evidence, stated that every even positive rational integer is a sum of two primes, for example $30 = 13 + 17$. All the available data seem to substantiate this wild guess in the dark. It has successfully resisted all analytical (and other) attacks. But the mere fact that modern analysis can take hold of and handle a problem of such inhuman difficulty is an indication of progress.

CHAPTER VIII

"STORMING THE HEAVENS"

TOWARD THE INFINITE

INFINITY and the infinite have long had a singular fascination for human thought. Theology, philosophy, mathematics and science have all at some stage of their development succumbed to the lure of the unending, the uncountable, the unbounded. "Only infinite mind can comprehend the infinite," according to one; "Cantor's doctrine of the mathematical infinite is the only genuine mathematics since the Greeks," according to another, while yet a third, contradicting both, declares that "the infinite is self inconsistent, and Cantor's theory of the mathematical infinite is untenable."

Here we reach a frontier of knowledge, and further progress will necessarily be slow. Some believe that mathematics is about to retrace many of the giant strides it made toward the infinite in the past half century; others foresee a steady progress in the direction already travelled.

The simple fact seems to be that no one at present can say exactly where mathematics stands with regard to its supposed conquest of the infinite, and no one can sensibly predict its future. Equally competent authorities hold diametrically opposing views.

With this caution against accepting anything in what follows as final, we may proceed to a short description of the kind of scaling ladders with which mathematicians "stormed the heavens," in Weyl's phrase, during the past fifty or seventy years.

HOW THE INFINITE ENTERED MATHEMATICS

The infinite entered mathematics early. Not to

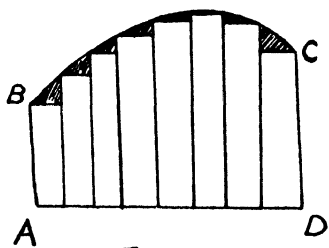

Figure 8

go too far back, let us glance at one type of problem which induced Archimedes in the Third Century B.C. to use the infinite in mathematics. It is that of finding the area under a curve, say the area ABCD. Cut the area up into strips of equal breadth, and disregard the shaded triangular bits. The remaining rectangles can easily be calculated.

Their sum is an approximation to ABCD. If by taking thinner and thinner rectangles the sum of the discarded shaded bits tends to zero, and if the sum of the rectangles tends to a *limit*, this limit is the required area. To reach the limit we must take the sum of an infinity of rectangles. This crude description must suffice.

With the invention of the calculus in the Seventeenth Century and its applications to the finding of areas, surfaces, and volumes of all imaginable shapes, such infinite summations, known as *integrations*, became one of the most powerful techniques of analysis.

Mathematical physics could not exist without integration. Consider for example the simple problem of calculating the work done as a *variable* force moves a body through a given distance, work being measured as force times distance in the proper units.

The process inverse to integration or summation is called *differentiation*. It will be sufficient here to state one geometrical application of differentiation. To draw a tangent line at a given point of a given curve necessitates the finding of the slope of the tangent line, and this is equivalent to performing a specific differentiation. Now consider this. It is *intuitively evident* that we can always draw a tangent to a continuous curve at a given point of the curve. Intuition is misleading; *there exist continuous curves which have no tangents at all.* We admit gladly that this is shocking to common sense, for it shocked mathematicians when Weierstrass first confronted them with such a curve in 1861.

Now let us go back to summation a moment. The solutions of multitudes of mathematical and physical problems lead to infinite sums. Here are three specimens:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\ldots;$$

$$1 + x^2 + x^4 + x^6 + \ldots\ldots;$$

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots\ldots.$$

These are almost pathologically simple, but they will do. The dots mean that the series are to continue *without end*, according to the law indicated in each case. Now, it can be proved that the first series *converges* to a definite, finite number as we proceed to infinity, adding and subtracting the fractions as they occur. If x is a real number, the second series converges only for such x as lie *between* -1 and $+1$; for all other real values of x the series *diverges*, that is, by adding a sufficient number of terms, the sum can be made to surpass any previously assigned number. The third series is divergent, although it does not look it. It seems incredible that the sum of a sufficient number of terms of this series can be made bigger than a billion billion billion, but such is the fact. Another astonishing thing is that the first series is *not* equal to $(1 + \frac{1}{3} + \frac{1}{5} + \ldots\ldots) - (\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots\ldots)$.

Is it not clear that if a physical problem, say the calculation of a temperature, yields as answer a *divergent* series, then that answer has no physical meaning? When such nonsense turns up we go back, revise our mathematics and reformulate the problem, or give it up.

One of the outstanding things young Abel and Cauchy did in the early decades of the Nineteenth Century was to provide the first methods whereby the convergence of a series can be tested.

From the foregoing handful of examples we can appreciate the program of that great triumvirate Weierstrass, Dedekind, and Cantor, who in 1859 to 1897 undertook a thorough examination of the mathematical infinite itself. Another impulse to an attack on the infinite was the problem of irrationals. What does the square root of two *mean*, if it is *not the ratio of any pair of whole numbers?*

Dedekind's attack on irrationals is a modern reverbration of Eudoxus. If either falls under the counter attack of modern skeptics, both fall. Paradoxical as it may seem, the last conclusion is no novelty of the Twentieth Century. Isaac

Barrow, the teacher of Newton, late in the Seventeenth Century acutely criticized Eudoxus, and Barrow's objections to the logic of the great Greel have been repeated by the leading Twentieth Century critics of the mathematical theory of the infinite elaborated by Weierstrass, Dedekind, and Cantor. If nobody listened to Barrow, the like cannot be said for Brouwer and his school.

Let us look at one or two of the central concepts of this controversial subject. Mathematical analysis—the calculus and every luxuriant growth that has sprung from its fertile soil in the past two centuries—derives its meaning and its life from the mathematical infinite. Without a firm foundation in the infinite, mathematical analysis treads at every step on dangerous ground.

COUNTING THE INFINITE

Let us consider first what counting means. At a glance we see that the two sets of letters x, y, z and X, Y, Z contain the *same number* of letters, namely 3. We say that two classes contain the *same number* of things if the things in both classes can be placed in *one-to-one correspondence*, that is, if we can *pair off* the things in the two classes and have none left over in either. For example, we can pair x with X, y with Y, z with Z. We say that two classes are *similar* if the things in them can be paired in one-to-one correspondence.

Observe this simple fact: the classes x, y, z, w and X, Y, Z are *not* similar. Try as we will, we cannot find a mate for *some one* of x, y, z, w. The reason here is plain; the first class contains *four* things, the second only *three*, and four is greater than three. Everyone saw this for thousands and thousands of years and, for a wonder, everyone saw straight. The next took genius of a high order to perceive. Georg Cantor (1845–1918) is the hero of this.

Consider *all* the positive rational integers

$$1, 2, 3, 4, 5, 6, 7, 8, \ldots,$$

and under each write its double, thus,

$$\begin{array}{l} 1, 2, 3, 4, 5, 6, 7, 8, \ldots \\ 2, 4, 6, 8, 10, 12, 14, 16, \ldots . \end{array}$$

How many numbers 2, 4, 6, ... are there in the second row? Exactly as many as there are numbers altogether in the first, for we got the second row by doubling the numbers in the first. The class of *all* the natural numbers 1, 2, 3, 4, ... is *similar to a part of itself*, namely to the class of all the even numbers 2, 4, 6, 8, ... *There are just as many even numbers as there are whole numbers altogether.*

This illustrates a fundamental distinction between finite and infinite classes. An *infinite class is similar to a part of itself; a finite class is similar to* *no part of itself.* "Part" there means *proper part,* namely, some but not all.

As another example, let us see that any two segments of a straight line contain the same number of points. (For brevity I am forced to omit many refinements which a mathematician would demand, but the following illustrates what is meant.) Suppose the segments AB and CD are of different

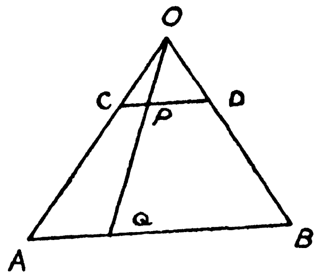

Figure 9

lengths. Place them parallel as in the figure, and let AC, BD meet in O. Take any point, say Q, on AB, and join OQ. Let OQ cut CD in P. This sort of construction puts the class of all points on AB into one-to-one correspondence with the class of all points on CD.

Is there no escape? What about *postulating* that the points on a line are not dense everywhere, but strung like dewdrops on a spiderweb, and that any segment contains only a *finite* number of points? Such finite, *discrete* geometries have been extensively investigated by American mathematicians in the past thirty years by the postulational method. But to say that space—whatever scientists and others mean by space—is granular in structure and not continuous is too repugnant to habit to be acceptable. Nevertheless, in physics, energy parted lightly enough with some of its continuity in 1900 when Planck quantized it, to avoid mathematical and physical absurdities. Instead of quantizing space, mathematicians at present prefer to overhaul their reasoning.

WHAT IS A NUMBER?

As analysis rests on numbers, I interpolate here an answer to the question of what a cardinal number is, say 2, or 3, or 4, or any other number which states "how many." The answer was given in 1884 by G. Frege, whose work passed almost unnoticed, possibly because much of it was written in an astounding symbolism which looked as complicated as a cross between a Babylonian cuneiform inscription and a Chinese classic in the original. It is the finest example of the precept that mathematicians should write so that he who runs may read. Bertrand Russell independently arrived at the same definition in 1901, and expressed it in plain English. Here it is:

The number of a class is the class of all those classes that are similar to it.

This is not meant to be simple. It is profound, and it is worth pondering until one grasps its truth intuitively. Beside this gem of abstract thought the visions of the mystics seem material and gross.

DEDEKIND'S CUT

How did Dedekind tame the irrationals? We postulated that the square root of 2 can be represented by a point on the line of all real numbers, lying somewhere between 1 and 2, and that, by approximating more and more closely we can narrow the interval in which the elusive number lies. But to trap it alone, and not get a whole brood of undesirables in the trap at the same time, requires supreme skill.

Dedekind provided this in his famous "cuts," which can be applied at any point of the line of reals. We need consider only that kind of cut which separates all the rational numbers into two classes of the following sort: each class contains at least one number: every number in the "upper" class is greater than every number in the "lower" class. Further, the numbers of the *upper* class have *no least number,* those of the *lower* class have *no greatest number.*

We can now imagine the "upper" and the

"lower" classes laid down on the line of real numbers. Owing to those provisos about no greatest and no least in the respective classes, the two classes will, as it were, strive to join one another. But they cannot, because any number in the upper is greater than every number in the lower. The place where they strive to join is the *cut*, and it defines *some irrational number*.

To locate the square root of 2 as a cut, we put into the *upper* class all those positive rational numbers whose squares are *greater than* 2, and into the *lower* class all other rational numbers. A moment's visualization will reveal that the elusive square root of 2 is definitely trapped between the two classes and is in the trap *alone*.

The Dedekind cut is at the root of modern mathematical analysis. Another root of that ever fertile tree is the vast theory of assemblages which, roughly, discusses among other things the properties of curves, surfaces, and so on, as *sets* or *classes of points*. An outstanding problem in the theory of sets is this: Can the elements in *any set whatever be well ordered?* For example, consider all the points on a segment of a straight line. Between any two points of the line we can always find another point of the line. How then shall we individualize this uncountable infinity of points and call each by its name according to any conceivable system of nomenclature? We do not know. A very famous postulate, Zermelo's of 1904, practically assumes that any assemblage can be well ordered, for his unaccepted proof rests on a doubtful postulate. The postulate asserts that if we are given any set of classes, each of which contains at least one thing, and no two of which have a thing in common, then there exists a class which has just one thing in each of the classes of the set. Why should this be true, if it is, of an *infinite* set of classes? This assumption, like all of the notions sketched in this chapter, has been challenged. It is much less innocent than it looks. We may have reached the great turning point in the progress of mathematics, and we may have to retrace our steps or swerve to one side to circumvent the unsurmountable. Whatever happens, we shall have lived through an epic age.

In the final chapter we shall indicate further difficulties.

CHAPTER IX

THE POWER OF ANALYSIS

TENDENCIES OF THE CENTURY

As WAS remarked of Africa, there is always something new coming out of analysis. This vast domain comprises everything that concerns continuously varying quantities. Its importance for natural science is therefore evident, since it is true, apparently, that "all things flow." Fixity is an illusion, and analysis gives us a firm grasp on the laws of continuous change.

The progress in analysis during the past century was beyond all precedent. Today its scope is so vast that probably no mathematician is competent in more than a province or two of the entire domain, Particularly is this so if, as seems legitimate, we include under analysis the modern developments of differential geometry—the investigation of geometrical curves, surfaces, and so on, from the study of configurations and structure in a small neighborhood. The last man to look out over the whole field of analysis was the universally-minded Henri Poincaré (1854–1912), and he was able to do so largely because great tracts of modern analysis were his own creations. On practically every department of mathematics this outstanding genius left his deep impression.

In all of this bewildering progress it is not easy to find commanding points of view from which to survey any significant expanse of the whole, unbounded territory. The boundaries in all directions are being pushed forward so rapidly that the eye soon loses them in the distance.

Nevertheless the past hundred years did indicate one or two general directions of advance, at which we must look. It may be said that three of the leading activities were the invention and exploitation of new species of functions in almost inconceivable variety, continual generalization, and drastic criticism of the foundations on which analysis rests.

Standards of rigor in proof were constantly raised. What had passed as satisfactory at an earlier period was minutely scrutinized, often found to be shaky, and firmly established according to the standards of the day. In this direction finality is not sought, for it is apparently unattainable. All that we can say is, in the words of a leading analyst, "sufficient unto the day is the rigor thereof."

Another tendency manifested itself. No sooner was a significant advance made in another department of mathematics than analysis seized upon the central ideas and assimilated them with voracious speed. Thus groups, invariants, much of geometry and parts of the higher arithmetic successively become its more or less willing prey. On the other hand, wherever it was found possible to apply the techniques of analysis to any other domain, whether purely mathematical or scientific, the advance was swift and sure.

CLERK MAXWELL AND EINSTEIN

Nowhere more strongly than in analysis do we appreciate the peculiar power of mathematical reasoning. This power is traceable, at least partly, to the fact that mathematics does not direct isolated or individual weapons at a problem, but unites whole complexes of subtle and penetrating chains of thought into new, intimately wrought engines of reason, often expressed by a single symbol whose laws of operation are once for all investigated, and then applies these *as units* to the problem on hand. It is somewhat like the advance of an entire, well coördinated army by a single order, instead of fussing over the details by which the individual companies are to manoeuvre. The mere creation of the single weapon begets unsuspected power in the parts of which it is composed and, operating as a unit, the whole achieves incomparably more than the sum of the achievements of the parts. Unsuspected possibilities present themselves automatically. Before the designer of the new weapon is aware of it, he has made a conquest of which he never dreamed.

Instance after instance of this peculiar power might be cited. We shall sketch only two briefly. In each it was not mathematics alone which won the victory. The insight, or intuition, of a great physicist was in each case necessary before the physical problem could be formulated mathematically. But neither advance could have been made—certainly neither *was* made—without powerful mathematical analysis. The ability to translate new scientific problems into mathematical symbols appears to be as rare as the genius which creates the mathematics to solve the problems.

Our first example goes back to 1864. In that year James Clerk Maxwell (1831–1879), having translated some of Michael Faraday's brilliant experimental discoveries in electromagnetism into a set of differential equations, and having filled out the set of equations to fit a physical hypothesis of his own, proceeded to manipulate the equations according to standard processes of mathematical analysis.

Now, one of the fundamental equations in mathematical physics expresses the fact that whatever satisfies the equation in a given instance is propagated throughout space in the form of waves. Moreover the equation contains the velocity with which the waves are propagated.

Manipulating his electromagnetic equations, Clerk Maxwell derived from them the wave equation of mathematical physics. The indicated velocity was that of light. Whether he was surprised at what the mathematics gave him, he does not record. At any rate he proceeded to exploit his discovery in grand fashion. He showed that electromagnetic disturbances must be propagated through space as waves. Further, from the manner in which the velocity entered the equation, he concluded that light is an electromagnetic disturbance.

This was in 1864. Clerk Maxwell died in 1879. In 1888, Heinrich Hertz (1857–1894), directly inspired by Clerk Maxwell's prediction of "wireless" electromagnetic waves, and guided by his predecessor's mathematics, set out to produce the waves experimentally and to determine their velocity. From his success has sprung the whole wireless and radio industry of today, and it all goes back to a few pages of mathematical analysis. But again we must emphasize that without Clerk Maxwell's extraordinary skill in *setting up the equations* and his physical intuition, the mathematics could not have got very far. On the other hand Hertz might never have even started.

Our second example goes back to 1854, when Bernhard Riemann (1826–1866) lectured before the venerable Gauss, "Prince of Mathematicians," "On the Hypotheses which lie at the Foundations of Geometry." This work pleased Gauss greatly, as it was a worthy sequel to his own of many years previously. One of Riemann's ideas concerned measurement in a curved space of any finite number of dimensions. In a curved two-dimensional space, for example the *surface* of a sphere, the formula of Pythagoras for the square on the longest side of a rightangled triangle, on which all everyday *measurements of distance are based,* does not hold. Riemann supplied a perfectly general formula, good for any of an infinity of "spaces" in which the curvature changes in any sufficiently general manner from point to point. Near the conclusion of his remarkable dissertation he made the following striking prediction of one great advance of the Twentieth Century.

"Either therefore the reality which underlies space must form a discrete manifold, or we must seek the ground of its metric relations outside it, in binding forces which act upon it.

"The answer to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain."

He goes on to say that narrow views and prejudice must not hamper the free investigation of all the novelties he has suggested.

Riemann's new geometry was one of the towering landmarks of the past century. A host of workers developed it, including E. B. Christoffel (1829–1900), after whom the famous "index symbols," familiar to physicists through relativity, are named. In all this work the theory of invariants played a leading part.

In the 1880's the geometer Ricci started a new development in Riemannian geometry. This was tied up with the concept of invariance. Ricci developed a calculus of extraordinary power for discovering those geometrical properties in Riemannian space which are invariant under extremely general (in fact almost any) transformations. This calculus is called now tensor analysis.

Consider now the statement of any physical fact or "law." If this statement contains essential references to the observer's particular way of expressing the law, then the supposed law is as much an expression of his tastes as of nature's. The point need not be labored. Einstein saw it (nobody else had till he pointed it out in 1915 as one of the cornerstones of his general relativity), and today it is appreciated by all who honor Einstein.

While Einstein was constructing his general theory in 1906 to 1915, he cast about for some calculus which would yield the differential equations of mathematical physics in *invariant form*. *Covariant* is the usual technical term, but in this connection it means the same. He found what he wanted in the calculus of Ricci. Einstein also needed an adequate geometry to describe the four-dimensional physical world of space-time. He found it in the work of Riemann and his successors. The rest was physics plus supreme genius, and is so well known today that it need not be repeated.

When general relativity first came out physicists were appalled at the unfamiliar mathematics—the commonplaces of over half a century to professional mathematicians. Today serious students of physics take all this in their stride and think no more of Christoffel symbols than they do of any other necessary mathematical tool. At the leading French technical school, l'Ecole Polytechnique, tensor analysis is taught along with mechanics in the second year of the regular course. It is far more practical than the older vector analysis.

Relativity has generously repaid its debts to geometry and analysis by starting a new golden age in geometry.

THE COMPLEX VARIABLE

A distinctive feature of progress in analysis during the past century was the stupendous development of the theory of functions of a complex variable. We saw earlier that if all the postulates of common algebra are retained, then no numbers more general than the complex satisfy the postulates. This gives a strong hint why functions of a complex variable sweep up so much of analysis. The development had well started by 1830; in fact Cauchy had then made his greatest contributions to this branch of analysis, of which he was the creator.

In the period we are considering two other ways of looking at the whole subject were discovered, one by Weierstrass, the other by Riemann.

Weierstrass *arithmetized* analysis. His universal tool was the *power series.* He regarded functions from the point of view of the convergent infinite series (like $a_0 + a_1z + a_2z^2 + \ldots\ldots + a_nz^n + \ldots\ldots$) which define them for values of the variable z in ranges appropriate to the functions.

Riemann on the other hand may be said to have *geometrized* the analysis of functions of a complex variable. By a most ingenious model of connected sheets or surfaces superimposed on a plane, for instance, he gave an intuitive picture of the properties of certain highly important complex functions, particularly those which take several different values for a given value of the variable. This development contributed greatly to *analysis situs*, the geometry which studies the properties of surfaces, volumes and the like which are invariant under a continuous group of transformations.

To give any adequate idea of this vast field in a few words is impossible, and we must pass on. Touching analysis situs however, we may mention one unsolved elementary problem which can be stated untechnically.

Practical map makers have never found a map, no matter how complicated, which cannot be colored, as a map should, by *four* colors. Prove that four colors are sufficient for *any* map in which contiguous countries have a line boundary in common. This looks easy. The solution may turn out to be extremely simple. The problem has important bearings on several others.

SPECIAL FUNCTIONS

Many of the most interesting functions which were intensively investigated from 1830 to 1900 were discovered from 1800 to 1830. Here again the field is too vast for more than a slight glance.

We shall mention only one type of functions which claimed the attention of analysts from about 1880 on.

Consider first any periodic phenomenon—say the passage of the tip of the minute hand of a watch over the 12-o'clock mark. This passage is made at regular intervals of one hour. We say that the position of the tip is a *periodic function of the time*, with *period* one hour. Periodic phenomena permeate science. Wave motion is an instance. For this reason, if no other, periodic functions were extensively investigated by the analysts of the past century and a half.

Expressing the periodicity in the above example algebraically, we write $f(t + 1) = f(t)$, which is read, "function of $t + 1$ is equal to function of t." Here "function" may be interpreted as position expressed in terms of time t. The thing to be noticed is that the *numerical value* of $f(t)$ is *unaltered* when we replace the *variable* t by the *linear expression* $t + 1$, depending only upon t. Thus the *value* of the function is *invariant under a particular linear transformation of the variable.*

Why stop here? Poincaré in the 1880's went much farther, and considered functions invariant under *groups* (in the technical sense explained previously) *of linear transformations of their variables.* The result was a new kingdom of analysis.

As a byproduct of all this, Poincaré *solved the general algebraic* equation of the nth degree by showing how its n roots can be expressed explicitly in terms of some of the functions he had created.

Finally, throughout the whole century, functions with any finite number of periods received much attention. These alone supply enough for a life's work.

GENERALIZATIONS

What is perhaps the most striking generalization originated in the work of Vito Volterra (1862–) and his school in the 1880's and 90's. In a word, Volterra investigated functions of a *non-denumerable infinity of variables*; a non-denumerable infinity being as many as there are of points on a straight line. For example, instead of looking at a curve as a relation between the coördinates of any point on it, we may consider *the curve itself* as the variable thing, and see what happens as one curve shades into another. The curve however, from another angle, is the set of all its points, and this set is non-denumerably infinite.

This, and what grew out of it, appears to be the true mathematical approach to all those physical problems where all the past history of a given thing has to be taken account of in predicting the future. For example, a steel bar when magnetized and then demagnetized, exhibits more or less permanent modifications which must be included in the mathematical analysis. Here and elsewhere, in economics for instance, the theory of *integral equations* and its modern extensions, largely the work of the past thirty years, is the clue. The subject originated with Abel and Murphy (who was a clergyman).

Roughly the distinction between such equations and those of the classical physics is this. In the classical mechanics and physics it is *rates of change* which enter the equations (*differential equations*); in the newer work it is the *inverses* of such rates, or *integrations* (infinite summations) which appear. From a given relation between these it is required to disentangle the functions which are integrated. Finally, in 1906, Henri Lebesgue revolutionized integration itself.

Those in a position to judge predict that these comparatively new fields will presently prove to be of an importance in science at least equal to that of differential equations, which have dominated physical science for over two centuries.

In differential equations the expansion during the past eighty years also has been enormous. Some of this was inspired directly by physics, much of it not.

BOUNDARY VALUE PROBLEMS

We alluded in the first chapter to boundary value problems. Such a problem is of the following type. Suppose we know the equation (as we do) which expresses the law of conduction of electricity in a medium, say in a sheet of copper of any shape. Applying the current at any parts of the sheet, we wish to know the subsequent distribution of electricity over the whole sheet. This is accomplished by making the *general* solution of the known equation satisfy the *initial* physical conditions. It is clear from the physics of the situation that once these conditions are given, say the places where the current is supplied and the amounts supplied there, the solution is uniquely determined. There cannot be conflicting distributions at any place at any time. Fitting of solutions of equations to prescribed initial conditions is technically known as solving boundary value problems.

This again leads to a vast field, still under vigorous development, in which many of the special functions devised by analysts in the past find their scientific interpretations.

CHAPTER X

BEDROCK

MATHEMATICAL EXISTENCE

To the uninitiated it may seem a very queer proceeding to build up vast systems of knowledge without seeing first whether the foundations will bear the superstructure. Mathematics did precisely that. As weaknesses began to appear in the foundations, and one part or another of the colossal edifice crumbled, mathematicians made hasty repairs and went on building, until more serious faults made themselves evident, and so on for well over a century.

Who shall criticize the builders? Certainly not those who have stood idly by without lifting a stone.

There is nothing reprehensible in the way mathematicians have worked. Any creative artist knows that criticism before a work is fairly complete is ruinou. . Only after the work is far enough along to be offered to the public is criticism relevant—when it cannot cause the artist to spoil his conception.

The critics of mathematics have been mathematicians almost without exception. The one reputable exception is Bishop Berkeley who, in the Eighteenth Century, showed that he knew what he was talking about when he gave the Newtonians a run for their money. In general the matters in dispute lie far below the surface, and are not likely to be observed by any but mathematicians as they go about their business.

In passing, let us record that Berkeley's specific criticisms were not met until the second half of the Nineteenth Century, when Weierstrass drove out of analysis the "infinitesimals," or "infinitely small quantities," of the Newtonians, to which Berkeley had so vigorously objected.

An anecdote concerning the arithmetically-minded Kronecker foreshadows one phase of the modern objections to mathematical reasoning. When everyone was congratulating Lindemann in 1882 over his proof that π is transcendental (see chapter VII), Kronecker said, "of what value is your beautiful proof, since irrational numbers *do not exist?*" Here Kronecker incidentally denied the "existence" of π, and he was less of a radical at that than some of the moderns.

What is the point at issue here? There are several. One which is disturbing mathematicians profoundly at present is this very question of what is meant by mathematical existence. We know—or used to think we knew—that with sufficient diligence (and stupidity) $\pi = 3.1415926\ldots\ldots$ could be calculated to an *indefinitely great number of decimals.* Indefinitely great? Not exactly; for who could ever do it? In what sense then, if any, does π "exist" as an infinite, nonrepeating decimal? I trust that I have not made this sound like a foolish quibble, for it is anything but that.

Kronecker said flatly that *unless we can give a definite means of constructing the mathematical things about which we talk and think we are reasoning, we are talking nonsense and not reasoning at all.* At one stroke he denied the validity of all the great work of the mathematical analysts on the infinite. To him it was worse than meaningless; it was useless.

There are those today who say Kronecker was right, and they cannot be silenced by an affection of superiority on the part of those who believe otherwise. There are equally strong men on both sides of the entire controversy.

Progress in this direction is being made by meeting Kronecker's objection step by step where it is important to do so, and actually exhibiting constructions for things that are used. It is impossible, of course, to meet fully any demand for a construction of an actual infinite; here we have to be content with exhibiting a process which, if carried out, would produce the required thing to any prescribed degree of accuracy.

I must warn the reader that the foregoing is an exceedingly crude description of extremely subtle difficulties, and that parts of the last sentence, if not all of it, would be regarded as sheer nonsense by one of the modern schools of mathematical thought. I can only suggest those profound problems and pass on to others, treating them equally sketchily.

The use of words alone in all these discussions is a treacherous proceeding. This also affects much of the technical literature on these disputes. Some mathematicians feel that if the ideas considered can not be adequately expressed in some appropriate symbolism, they are too dangerous to be handled. The history of philosophy is a sufficient warning.

HILBERT'S LOGIC

So great is the average mathematician's distrust of purely verbal arguments that Hilbert, beginning about 1925, proposed that for the present at least mathematicians forget about the "meanings" of their elaborate game with symbols, and concentrate their attention on the game itself. He and his pupils have formulated the rules of play in an unassuming *theory of demonstration*, whose aim is to *prove* that mathematics is *free of contradiction*.

The rules are expressed in symbols with brief verbal instructions for their use, and are a strikingly simple form of symbolic logic. Hilbert assumes as known the logical *and*, *or*, *not*, *if-then*, and some more mathematical notions, equally elementary. For example, *not*-X is written X; X & Y is X *and* Y, etc., and a typical rule of play permits us to put $X\&Y$ for $Y\&X$ if the latter appears in shifting the "meaningless marks" about in accordance with the rules.

The object of all this is to *prove* that the innocent looking rules will never lead to a contradiction.

Critics of the movement deny that it has any significance for the points in dispute. Some of them further deny the modest claim made by Hilbert's adherents that they have proved the consistency of mathematics up to the point where only a *finite* number, or only a *countable infinity* of elements are concerned. A countable infinity, we recall, is as many as there are of *all* the natural numbers 1, 2, 3,....... That wretched monosyllable "all" has caused mathematicians more trouble than all the rest of the dictionary.

That Hilbert's method has established the consistency of the Dedekind cut, of Cantor's theory of the infinite, of any of the theory of sets, or of mathematical analysis, is not claimed even by its most ardent partisans.

The outstanding merit of Hilbert's contribution is its fearless exposure of the weak spots in mathematics and the attention which it commands for these spots from competent professional mathematicians. That a man of Hilbert's mathematical eminence should put the supreme effort of his great career into this crisis is a sufficient reply to those who belittle the honest if disturbing strictures of the critics.

FURTHER DIFFICULTIES

Most of the paradoxes which mathematics is striving to resolve entered with the mathematical infinite. This led to a critical examination of mathematical reasoning of any type. From this the criticisms have reached out to the classical logic of Aristotle, which for over two thousand years reigned free of suspicion that it might not be universally valid.

If Aristotle ever heard of an infinite set in the sense of mathematics, he seems to have left no record of the fact. What reason can be given that Aristotelian logic applied to infinite sets will not produce contradictions? None whatever. In 1906 Henri Lebesgue, who revolutionized the theory of integration, stated explicitly that he was not convinced that a statement about an infinite set is necessarily true or false. In other words there may be a *third* possibility, between truth and falsity, or there may be nothing but nonsense in any assertion about an infinite set.

The reader may amuse himself by picking the foregoing sentence to pieces in the light of the very objection it raises, namely to the universal validity of the law of excluded middle—an assertion is true or it is false. The sentence is riddled with inconsistencies. It is a fair sample of the difficulty of talking sense about the fundamentals of reasoning—mathematical or other.

A leading contention of the strong intuitionist school led by L. E. J. Brouwer is that the logic of Aristotle is partly inappropriate for mathematics. In particular, the law of excluded middle is not always admissible, and Euclid's method of indirect proof is not free from very serious objection. One aim of this school is to revise mathematical reasoning so as to avoid the disputed points. It is astonishing to see how far we can go in this direction.

What is desired in this: to weed out what can be shown to be definitely erroneous in mathematics, and to root what remains in uninfected soil. Further, if the whole critical movement is not to be utterly barren, it must account for the undisputed fact that mathematical reasoning has led to results which, by common consent, are *true*, whatever "truth" may mean. If the reasoning by which correct results were reached was irreparably wrong, then that in itself will be an astounding and far-reaching discovery.

As the reader may be interested in seeing the kind of puzzle which is worrying mathematicians, I shall transcribe a simple one for him to ponder. It is known as Russell's paradox. This particular one is cited because Russell with A. N. Whitehead in 1910–1914 produced the monumental *Principia Mathematica* which aimed, among other things, to resolve the paradoxes of analysis and the theory of sets. This gave a new impulse to mathematical rigor which has lasted to the present day. Here is the paradox, not yet resolved, as the machinery which Russell contructed for such purposes has been abandoned by mathematicians.

"Let w be the class of all those classes which are not members of themselves. Then, whatever class x may be, x is a w is *equivalent* to x is not an x. Hence, giving to x the value w, 'w is a w' is equivalent to 'w is not a w'.

Two propositions are called *equivalent* when both are true or both are false." (*American Journal of Mathematics*, vol. 30, 1908, p. 222.)

It was the appearance of several similar paradoxes in mathematical analysis in the past forty years that led to the present upheaval.

TO OUR SUCCESSORS

After the splendid achievements of the Century of Progress in mathematics, it seems ungracious to close on a note of doubt. The sentiments of creative mathematicians cannot be disregarded. Surely their feeling for what is true in mathematics is not without significance. Almost without exception these men feel this about the past and probable future of their beloved mathematics: not *all* of those giants of the past can have been fools *all* of the time, and we may rest assured that greater shall come after them.

Wisdom was not born with us, nor will it perish when we descend into the shadows with a regretful backward glance that other eyes than ours are already lit by the dawn of a new and truer mathematics.