Using a Fractal Program to Model Vortical Flows

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Abstract

While extraordinarily complex in their physics, vortices have always been a part of human culture. This work presents a technique for simulating vortical flows using a fractal program on a personal computer. The resulting flows model real flows in some respects and demonstrate periodicity, chaos, and fractal structures. Also, the aesthetic interest of vortical flows introduces an opportunity for the artist to go beyond the spirals typically found in fractal art.

Introduction

Vortices have long been subjects of human interest. The flow of fluids (gases or liquids) through and around vortices is a subject as old as science. The vortex and its spiral cousin have played significant roles in the symbolism and mythology of various peoples throughout history, as chronicled by Lugt [1].

In modern science, vortices are seen as a fluid mechanics phenomenon, simultaneously simple and complex. A typical occurrence of vortices is the wing-tip vortex coming off of each wing of an airplane, wonderfully illustrated in Figure 1 [2].

Figure 1: *Wing tip vortex [2].*

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t)$$

Figure 2: *Simplified Navier-Stokes equations.*

The rules governing fluid flow, the Navier-Stokes equations, have been known for well over a hundred years, but are sufficiently complicated that a $1 million "reward" has been offered for the solution of the simplified version shown in Figure 2 [3]. Indeed, this complexity is one reason why solving fluid flow equations is an activity in which supercomputers are frequently employed.

The current work was motivated by a desire to work with vortices on a more reasonable scale—the personal computer. The computing power of a typical desktop computer is sufficient to allow the layperson to study vortices scientifically and artistically, provided she had the requisite software.

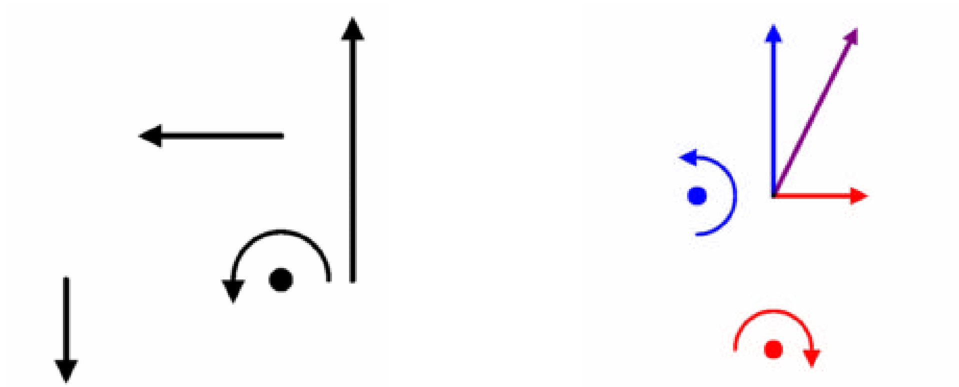
Potential Flow

The Navier-Stokes equations apply to flows that: change in time, vary in all three spatial dimensions, contain viscosity, and may contain shock waves. By eliminating those effects from consideration, the scenario is reduced to "potential flow," so called because the velocity can be specified using a potential function [4]. Solutions to potential flow are particularly simple, yet can still be used to model real situations.

One solution of interest is the point vortex. It is a point at which the equations break down, but around which the fluid rotates like a planet orbits a star. In particular, the velocity around the point is given by:

$$v = K/r,$$

where K is the strength of the vortex and r is the distance from the point. Such a velocity field is shown in Figure 3a, where the length of the arrow corresponds to the speed of the particle at that point.



a) *Velocity around a single vortex.* **b)** *Velocity components and resultant vector.*

Figure 3: *Velocity induced by point vortices.*

If a system of multiple vortices is created, then every vortex asserts its influence on every other point, as shown in Figure 3b. At each point, the resulting velocity is the sum of the velocity components induced by all the vortices, and the fluid particle moves with that velocity. Each vortex generates a velocity at the location of the other vortices, so the entire system moves in its own self-induced velocity field.

Computer Implementation

Conceptually, determining the motion of a particle in a vortex field is fairly straightforward. First, the vector sum of the velocities induced at the particle by the vortices is determined. If the particle is itself a vortex, then it is assumed to have no effect on itself. If a particle is too close to a vortex, then the induced velocities can become large, due to the 1/r behavior of the velocity. Consequently, a core region is utilized, such that the velocity falls to zero at the point of the vortex, and increases to the potential flow value at the edge of the core.

Once the total induced velocity is determined, it is integrated to yield a change in the particle's position. This can be done in a variety of ways; the present work used a fourth-order accurate Runge-Kutta scheme [5] for its accuracy, stability, and relative ease of implementation. Other techniques, such as forward Euler, may be used, and represent tradeoffs between speed, accuracy, and ease of use. They could prove useful for studying the effects of algorithm choice and for creating interesting images.

The computing platform for this work was the shareware program Ultra Fractal [6] running on Microsoft Windows. While not designed as a flow solver, Ultra Fractal was suitable for the task, due to its general structure. Conceptually, the program works likes this for every pixel in the image window:

1. Determine the pixel's coordinates.
2. Optionally transform the coordinates to a new value.
3. Input the (transformed) pixel value to a function that is iterated.
4. Feed the iterated function values to a coloring algorithm, which outputs an index value.
5. Interpolate a color palette with the index value to determine the pixel's color.
6. Repeat steps 1-5 for optional additional layers, and combine the layers into a final image.

Steps 2, 3, and 4 use user-specified third-party formulas, of which there are hundreds [7]. This allows Ultra Fractal to be used as a standard fractal generator, a vector-based drawing program [8], or an engine for solving dynamical systems problems, as in the present case. The iterative nature of the program means that each iteration can be considered a time step, advancing the solution in time as additional iterations are carried out.

Three solution types were used to investigate periodic and chaotic vortical flows. In the first, the paths of the individual vortices were drawn. This was useful for determining the qualitative nature of the system, e.g., periodic. The second type involved placing passive markers in the flow and drawing their paths. These markers could either be considered to be vortices of zero strength, or their velocities could be determined by the (stored) vortex locations. The third solution type was one in which a continuous region of points was transformed by the vortex field, and its final position shown. This was analogous to watching a drop of cream diffuse into a cup of coffee that had been stirred by a spoon.

Periodic Motions

It is well known that a series of point vortices spaced equally on a circle will rotate on that circle, if the vortices all have the same strength and rotation. The simple case of two vortices rotating around their common center is illustrated in Figure 4. Here, the rendering is done in terms of the distortion of the fluid surrounding the vortices, from the perspective of the vortices. The original field was gradated from black to white to black again, in terms of the angle around the center point (center of the image). Three snapshots show the field as the vortices drag fluid particles further and further away from their starting points. Time is given in terms of fractions of a period; one period being the time needed for each vortex to complete one circle. At a time of 0.01 period, the original is largely undisturbed. The two vortices can be seen as the centers of the two gentle spirals, above and below the center of the gradient. By 0.1 period, the spirals are becoming more defined, and a boundary is appearing between fluid that was originally on one side of the field versus the other side. After a full period has progressed, the boundary is very sharp in the outer field, and is being wound into a fractal in the inner field surrounding the vortices.

a) *Time = 0.01 period.* **b)** *Time = 0.1 period.* **c)** *Time = 1 period.*

Figure 4: *Distortion of flow field by 2 rotating vortices.*

If the rotation of the one vortex is reversed, the pair now translates at a constant speed, each vortex inducing the same velocity on the other. While this motion may seem trivial, the motion of the particles is not. The imaginary line between the vortices forms a wall, in that no particles move across it. Those on the top side of the wall orbit the upper vortex, likewise those on the bottom half. The location of the particle determines how long it takes to orbit; the closer particles have higher velocities and orbit is much less time. Figure 5 shows four traces for particles nearer to the wall (bottom of the image) to close to the upper vortex (just below the middle of the image). In the time it takes for the bottom-most tracer to complete one orbit, the subsequent tracers complete two, four, and eight orbits, respectively. The motion of the vortex is a horizontal line through the loops.

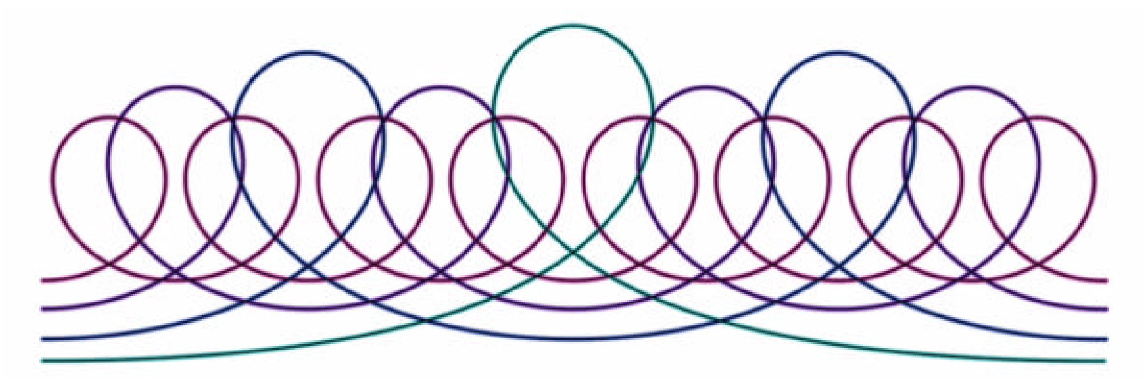


Figure 5: *Motion of passive markers around a translating vortex.*

The motion induced by four rotating vortices is shown in Figure 6. Here, the vortices were placed at the corners of a square. Since they all had the same strength and rotational orientation, they induced the same velocity at every corner point and the vortices retained their equal spacing. A black circle covering the four vortices was superimposed on the fluid, and the images detail the deformation of the circle with time. The rollup of the circle into the spirals is visually compelling and suggests that these flows contain fractal attractors to which particles will be drawn, and basin boundaries across which particles may not pass. In this case, the fluid in the center remains there and does not diffuse into the outer field.

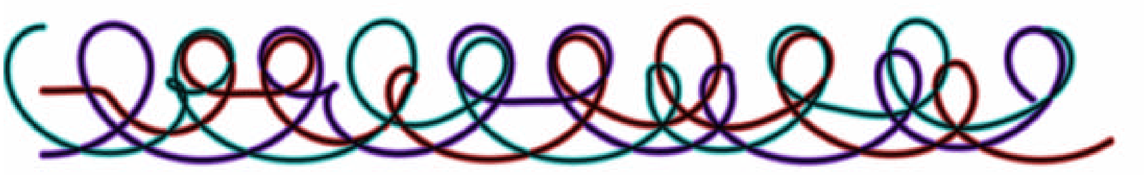
a) *Time = 0.01 period.* **b)** *Time = 0.1 period.* **c)** *time = 1 period.*

Figure 6: *Deformation of a circular spot by four vortices.*

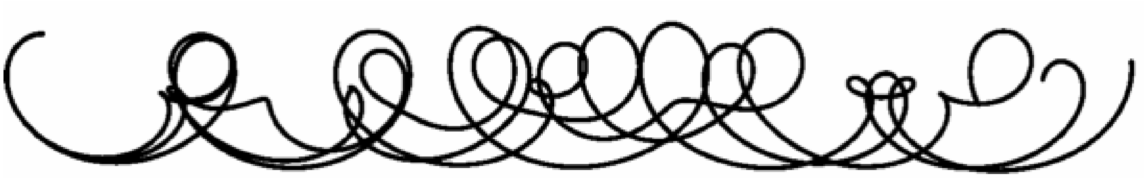
Chaotic Flow

When the system includes three or more vortices, then chaotic flows are possible [9]. Here, two particles that begin close together will move away from each other at an exponential rate. Alternatively, small changes in numerical parameters of the simulation will lead to large differences in the details, although the statistical behavior remains the same.

Chaotic flows were investigated using the example of six vortices representing three vortex rings. A vortex ring can be thought of as a point vortex that has been extruded into a line and then had its ends joined into a ring. A pair of point vortices can serve as a two-dimensional analog to the ring if both point vortices have the same strength but rotate in opposite directions. The line between the points represents the axis of the ring. Consider a system of two rings, whose axes are along the same line, moving in the same direction. As shown experimentally in [10], the rings will execute a motion called, "leapfrogging." If ring A begins behind ring B, then A will compress, shoot through the center of B, then expand and slow down. Then B will do the same, moving through and ahead of A, and the two will move downstream together, regularly leapfrogging each other. With a system of three or more rings, the leapfrogging is generally chaotic, with no set pattern of which ring will move through another, nor no set time interval between passings.



a) *Paths of vortices representing the tops of the rings.*

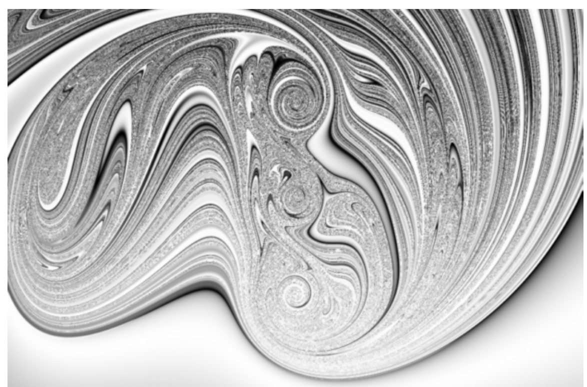


b) *Paths of the top vortex for three different time steps.*

Figure 7: *Simulation of chaotic leapfrogging with three ring vortices.*

Figure 7 shows a simulation of this motion for three rings. In the upper panel, each trace shows the path of a single vortex representing the top of a ring. As the trace moves upward in the image, the ring would be getting larger. The overall direction of motion is from left to right. Clearly demonstrated is the lack of any regular looping structure in the motions of the vortices. The bottom panel presents this phenomenon a bit differently: each trace represents the path of the top vortex, similar to the path beginning in the upper left corner of Figure 7a. The difference between the traces in 7b is the location of the first vortex. Three slightly different postions were used, a baseline, 1/1000 of a unit to the left, and 1/1000 unit to the right. The initial vertical distance between the top and middle vortices was 10 units. The traces begin together at the left, but rapidly diverge and have no correlation at the right of the image.

As might be expected, if the vortices have chaotic motions, then the particles in the field would also have chaotic motions. This is illustrated in Figure 8. Five passive markers were inserted into the fluid near the original vortex locations (lower left corner of Figure 8a). As time progressed, the particles' paths diverged quickly and the paths fill the image. The elapsed time is half of that in Figure 7. That the markers moved chaotically is suggested by the fractal structure of the flow field, shown in Figure 8b. In the region around the vortices, the fluid is warped and wrapped around like taffy. The black and white striations in the figure show how particles that began on opposite sides of the field wound up next to each other. Conversely, particles that began close to each other wound up far apart after very little time.

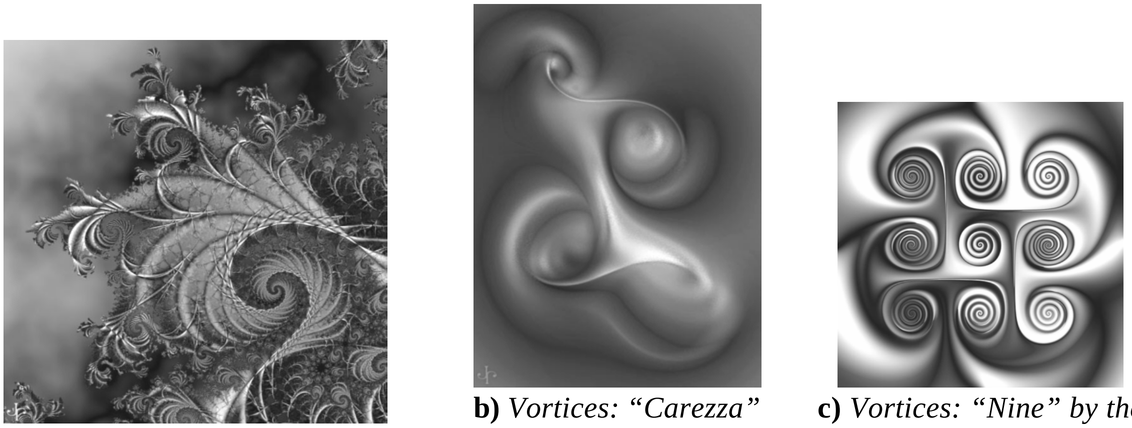


a) *Paths of markers in chaotic motion.* **b)** *Fractal distortion of field around vortices.*

Figure 8: *Field around three chaotic leapfrogging vortex rings.*

Aesthetics

Vortices and spirals have long been employed as artistic motifs, in cultures that were widespread geographically and chronologically [1, 11]. Currently, spirals are enjoying popularity as a key in fractal art. While more difficult to implement, vortices are also being used artistically. Examples of both can be found in the work of Janet Parke [12] and are shown in Figure 9. Also included in the figure is a piece created by the author, using the present methodology. Indeed, several of the figures in this paper have an aesthetic appeal to them, and the interested reader is invited to see what they can create with these ideas.



a) *Spirals: "Taupensky" (1999).* **b)** *Vortices: "Carezza" (2000).* **c)** *Vortices: "Nine" by the author (2003).*

Figure 9: *Spirals and vortices used in fractal art.*

Conclusions

Vortical flows have played a significant role in the culture of humankind, in both scientific and non-scientific ways. The equations governing fluid vortices can be simplified to the extent where they can be feasibly solved numerically on a personal computer. Even though these simple solutions lack many of the characteristics of real fluid flows, they are accessible, complex enough to be interesting, and often beautiful. Judicious choices in the creation of the vortex system can lead to flows containing periodicity, chaos, and fractal structures.

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